1) ODE
2) 1-D phase portraits
3). solve autonomous 1d ODE
(& more general separable ODEs)

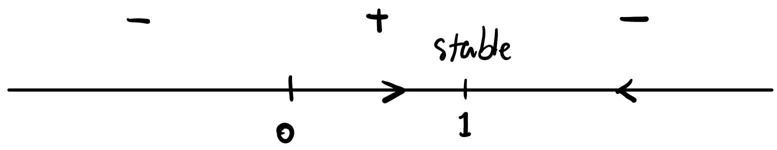
Def. (Take 1) An ODE of first order in $\mathbb{R}^d$ is a relation of the form $\dot{x}(t) = v(t, x(t))$, where $v: \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a vector field (function)

phase space = all states of the system = $\mathbb{R}^d$

To solve an equation $(*)$ on a time interval $I = (a,b) \subset \mathbb{R}$. means to find all functions $x: I \longrightarrow \mathbb{R}^d$ satisfying $(*)$ on I.
These functions $x()$ are called solutions, trajectories, interval curves.
There're some analytic solutions, but often we cannot obtain solutions explicitly.

Phase portraits of 1-order autonomous ODEs in 1-D.
Def. If $v(t,x) = v(x)$ for all t, x, then the ODE is called autonomous (no external influences) + doesn't show explicitly.
Phase portraits are 

EX. Logistic Equation: $v(x) = x - x^2$ | sets a cap to # of fish
Fish farm, x = thousands of fish
$\dot{x} = x$ would mean exponential growth. $x(t) = x(0)e^t$
Critical Points: values of x where $v(x) = 0$.
$x - x^2 = 0 \Rightarrow x = 0, 1$



$x(t) \equiv 0$ $\qquad\qquad$ $x(t) \equiv 1$

Fish Harvesting

Harvest at rate of cx, $c > 0$

Find c that guarantees the best long term profit.

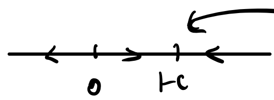
$\rightarrow \dot{x} = x - x^2 - cx = x(1-x-c)$

Phase Portrait: Critical points are 0 and $1-c$.

If $c > 1$, decays to 0 quickly.

If $c = 1$, always decreasing.

If $0 < c < 1$,



stable equilibrium $\quad$ Long term population $\approx 1-c = x_{stable}$

Catch rate at equilibrium is $cx_{stable} = c(1-c)$, $\max c(1-c) = \frac{1}{4}$, $c = \frac{1}{2}$

Solving autonomous 1-D ODEs.

$\dot{x} = v(x)$, $\dot{x}(t) = v(x(t))$

E.x. Solve $\dot{x} = x$.

$\frac{dx}{dt} = x \qquad \frac{dx}{x} = dt \qquad \int \frac{dx}{x} = \int dt \qquad \ln|x| + C_1 = t + C_2$

$\ln|x| = t + C_3 \qquad |x| = e^{t+C_3} = C_4 e^t. \qquad x = Ce^t, \ C \in \mathbb{R}$

$\frac{\dot{x}(t)}{x(t)} = 1 \qquad \frac{d}{dt}(\ln|x|) = 1. \quad \ln(x) = t + c$

In general, to solve $\frac{dx}{dt} = v(x)$:

(1). find all critical points of x s.t. $v(x)=0$. If x_0 is critical, then $x(t) \equiv x_0$ is a solution.

(2). Between critical points, rewrite $\frac{dx}{v(x)} = dt$

Find $A(x) = \int \frac{dx}{v(x)}$, write $\boxed{A(x) = t + C. \quad (**)}$ separation of variables

(2') Equivalently, write $\frac{x'(t)}{v(x)} = 1$

Find solution $A(x)$ s.t. $\frac{d}{dt} A(x(t)) = \frac{\dot{x}(t)}{v(x(t))}$, $\quad A'(x) = \frac{1}{v(x)}$

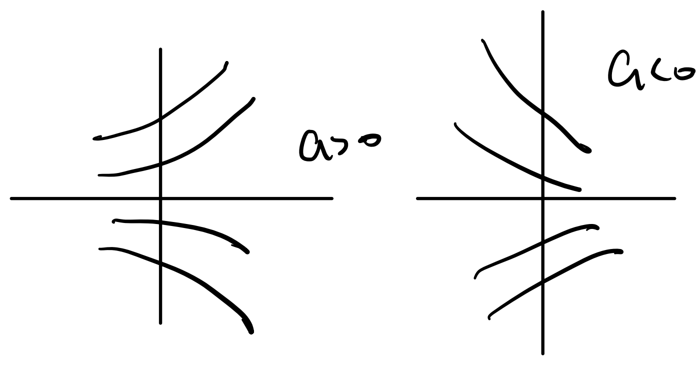
$$\| \\ A'(x(t)) \cdot \dot{x}(t)$$

(3). Solve $(**)$ for x

$x = A^{-1}(t+C)$ choose continuous branches of x.

More examples:

$\dot{x} = ax \qquad x(t) = x(0)e^{at}$

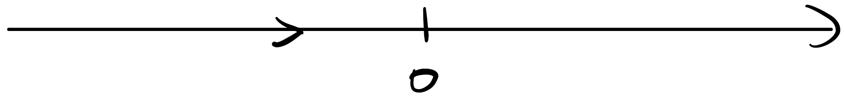


$\dot{x} = x^2 \qquad \frac{dx}{dt} = x^2 \qquad \int \frac{dx}{x^2} = \int dt \qquad -\frac{1}{x} = t + C$

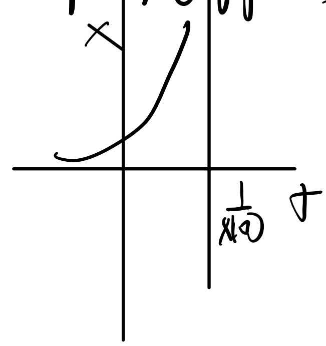
$x = \frac{-1}{t+C}$ Find C in terms of $x(0)$.

Plug in $t=0$ into solution, $x(0) = \frac{1}{-C}$, $C = -\frac{1}{x(0)}$

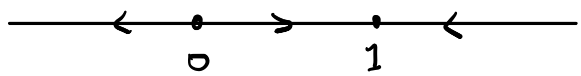
$$x(t) = \frac{x(0)}{1-x(0)t}$$

Phase portrait 

If $x(0)>0$, A singularity or blow up happens when $t \uparrow$



Fish harvest revisit: logistic equations

$\dot{x} = x - x^2$ 

$$\int \frac{dx}{x-x^2} = \int dt = t + C = \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx$$

$$= \ln|x| - \ln|x-1|$$

$$\ln\left|\frac{x}{x-1}\right| = t + C$$

$$x - x^2 = \pm Ce^{t}$$

$$\frac{x}{x-1} = Ce^t$$

$$x = Ce^t(x-1)$$

$$(1-Ce^t)x = -Ce^t$$

$$x = \frac{-Ce^t}{1-Ce^t}$$

$$x = \frac{Ce^t}{Ce^t-1}$$

Solutions are called logistic curves

9/19 <u>Solve a few more ODEs.</u>

If an ODE can be rewritten as $\dot{x}(t) = a(x)\,b(t)$, it's called separable.

$$A(x,t)\dot{x}(t) + B(x(t),t) = 0.$$

Separation of variables.

$$\frac{dx}{dt} = a(x)\,b(t) \Rightarrow \int \frac{dx}{a(x)} = \int b(t)\,dt.$$

Find $A(x) = \int \frac{dx}{a(x)}$, $B(t) = \int b(t)\,dt$.

$\Rightarrow A(x) = B(t) + C$

$x(t) = A^{-1}(B(t) + C)$

E.x. $\dot{x} = \frac{t^2}{x}$ not autonomous

near $(t_0, x_0) = (1, 1)$.

(i.e., $x(1) = 1$

$$\frac{x^2}{2} = \frac{t^3}{3} + C, \quad x = \pm\sqrt{\frac{2t^2}{3} + 2C}$$

plug in $x(1) = 1$ to determine the sign. $+$

So, $x(t) = \sqrt{\frac{2t^3}{3} + \frac{1}{3}}$

Homogenuous equations:
change of variables

$$\dot{x} = F(t, x) = \varphi\left(\frac{x}{t}\right) = F(\overset{\text{any constant}}{ct, cx})$$

Substitution:

$u = \frac{x}{t}$, $x = ut = u(t)\, t$

$\dot{x} = \frac{d}{dt}(u(t)\, t) = \dot{u} t + u \cdot 1$

$\dot{x}(t) + a(t)\, x(t) = 0$

homogeneous linear ord 1.

?

$\varphi(u) = \dot{u} t + u$

$\dot{u} = \frac{\varphi(u) - u}{t}$ new seperable ODE equation for u.

Ex $\dot{x} = \frac{t+x}{t} = 1 + \frac{x}{t}$, homegenuous

$u = \frac{x}{t}$, $x = ut$

lhs $\dot{x} = \dot{u} t + u = 1 + u =$ rhs

$\dot{u} t = 1$, $u = \ln|t| + C$.

$x = (\ln|t| + C)\, t$, near (t_0, x_0)

Homogenous linear non autonomous eq. of order 1.

$\dot{x} + a(t)x = 0$. $\leftarrow$ separable 0 is always a solution.

Linearity: 1). if x & y are 2 solutions then $z(t) = x(t) + y(t)$ is also a solution.

$$\dot{z} + a(t) z = \dot{x} + \dot{y} + a(t)(x+y)$$
$$= (\dot{x} + a(t)x) + (\dot{y} + a(t)y)$$
$$= 0$$

2). $z(t) = c\,x(t)$ is also a solution

$\frac{\dot{x}}{x} = -a(t)$, $\ln|x(t)| = u$ (change of var)

$\dot{u} = -a(t)$, $u = -\int a(t)\,dt + C_1$

$x(t) = \pm e^{C_0} e^{-\int a(t)dt}$

$$\boxed{x(t) = C e^{-\int a(t)dt}}$$

$x_0 e^{-\int_{t_0}^{t} a(s)\,ds}$

general solution

If $a(t) \equiv A = \text{const}$

$$\boxed{x(t) = Ce^{-At}}$$

$\int A\,dt = At$

Non homogeneous linear eqn.

$\dot{x} + a(t)x = b(t)$ (1)

Variation of constants.

It turns out one can always find function $c(t)$ such that $x(t) = c(t)\underbrace{e^{-\int a(t)dt}}_{y(t)}$ (2) is a solution.

Combine (1), (2). Plug $x(t) = c(t)\,y(t)$ into $\dot{x} + a(t)x = b(t)$.

$$\dot{c}(t)\,y(t) + \underbrace{c(t)\,\dot{y}(t) + a(t)\,c(t)\,y(t)}_{c(t)\,\underbrace{(\dot{y}(t) + a(t)\,y(t))}_{=0 \text{ since linearity!!}}} = b(t)$$

$$\dot{c}(t)\,y(t) = b(t)$$

$$\dot{c}(t) = \frac{b(t)}{y(t)} \qquad c(t) = \int \frac{b(t)}{y(t)}\,dt + C.$$

Conclusion:
$$x(t) = \left(\int \frac{b(t)}{y(t)}\,dt + C\right) y(t)$$
$$= y(t)\int \frac{b(t)}{y(t)}\,dt + C\,y(t).$$

$x(0) = x_0$, $x(0) = C\,\underbrace{y(0)}_{=1}$

$C = x_0$.

$$x(t) = C e^{-\int_0^t a(s)ds} + \int_0^t \frac{y(t)}{y(s)}\,b(s)\,ds$$

$$\frac{y(t)}{y(s)} = \frac{e^{-\int_0^t a(r)dr}}{e^{-\int_0^s a(r)dr}} = e^{-\left(\int_0^t a(r)dr - \int_0^s a(r)dr\right)} = e^{-\int_s^t a(r)dr}$$

$$\boxed{x(t) = x_0\, e^{-\int_0^t a(s)ds} + \int_0^t b(s)\, e^{-\int_s^t a(r)dr}\,ds}$$

(H) $\dot{x}(t) + a(t)x(t) = 0.$ $\qquad \frac{dx}{x(t)} = a(t)dt.$

(N) $\dot{x}(t) + a(t)x(t) = b(t)$

Solve H.

$x(t_0) = x_0.$ $\quad \int a(t)dt = \int_{t_0}^{t} a(s)ds \qquad Ce^{-\int a(t)dt}.$

$x = Ce^{-\int_{t_0}^{t} a(s)ds}$. Plug in t_0, x_0.

$\underline{x_0 = C}$

$$\boxed{x(t) = x_0 e^{-\int_{t_0}^{t} a(s)ds}}$$

$$\int a(t)dt = \int_{t_0}^{t} a(s)ds$$

(N) $Ax = b.$ $\qquad x_H$ solves (H)

x_N solve (N).

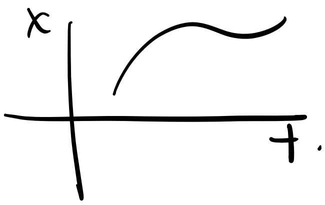
(H) $Ax = 0$ $\qquad x_H + x_N$ solve N.

$A(x_N + x_H) = b + 0 = b.$

Exact equations:

$a(t,x)\dot{x}(t) + b(t,x) = 0$. and there's a function $\Phi(t,x)$

s.t. $\frac{\partial}{\partial t}\Phi(t,x) = b(t,x)$ $\quad \frac{\partial}{\partial x}\Phi(t,x) = a(t,x)$.



$$\frac{d}{dt}\Phi(t,x(t)) = \partial_t\Phi(t,x(t)) + \partial_x\Phi(t,x(t))\,\dot{x}(t).$$

$$= b(t,x(t)) + a(t,x(t))\,\dot{x}(t)$$

x is a solution $\Rightarrow \frac{d}{dt}\Phi(t,x(t)) = 0$.

so $\Phi(t,x(t)) = C = \text{const}$.

Solve for x, obtain a solution

Ex. $3x^2(1+t^2)\dot{x} + 2tx^3 = 0$

This's an exact solution with $\Phi = x^3(1+t^2)$.

$$x^3 = \frac{C}{(1+t^2)^{\frac{1}{3}}}$$

Critirion for correctness:

THM: $\exists\, \Phi$, $\quad \partial_t \Phi(t,x) = b(t,x)$

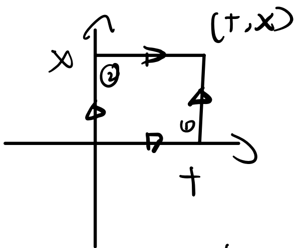
$$\partial_x \Phi(t,x) = a(t,x)$$

$$\Longleftrightarrow \quad \partial_t a(t,x) = \partial_x b(t,x)$$

Proof. $\Rightarrow$ $\partial_t a = \partial_t \partial_x \Phi = \partial_x \partial_t \Phi = \partial_x b.$

switch

$\Leftarrow$ Define $\Phi_1(t,x) = \int_0^t b(s,0)\,ds + \int_0^x a(t,y)\,dy.$ Along ①.



$$\Phi_2(t,x) = \int_0^x a(0,y)\,dy + \int_0^t b(s,x)\,ds.$$

Claim: $\Phi_1 \equiv \Phi_2$. Stokes Theorem

$$\Phi_1 = \Phi_2 = \left(\int_0^x a(t,y)\,dy - \int_0^x a(0,y)\,dy\right) - \left(\int_0^t b(s,x)\,ds - \int_0^t b(s,0)\,ds\right) = I_1 - I_2.$$

$$I_1 = \int_0^x \big(a(t,y) - a(0,y)\big)\,dy = \int_0^x \int_0^t \partial_t a(s,y)\,ds\,dy$$

$$I_2 = \int_0^t \int_0^x \partial_x b(s,y)\,dy\,ds$$

Fubini Thm.

$$\partial_x \Phi = \partial_x \Phi_1 = a(t,x)$$

$$\partial_t \Phi = \partial_t \Phi_2 = b(t,x) \quad \blacksquare$$

Existence and Uniqueness of solutions:

Solution is "nice" if $F(t,x)$ of $\dot{x} = F(t,x)$ is a Lipschitz function. Lipschitz implies uniform continuous.

<u>Def</u> $F: \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$

$(t,x) \longmapsto F(t,x)$

is Lipschitz in x if $\exists\, C > 0$ such that for all $x, y \in \mathbb{R}^d$ for all times $t \in \mathbb{R}$

$|F(t,x) - F(t,y)| \leq C\,|x-y|$, where $|\cdot|$ is Euclidean norm.

Thm. Suppose $F(\cdot,\cdot)$ is Lipschitz in its second argument. Then for any time interval I containing 0, for any initial condition $x_0 \in \mathbb{R}^d$, there's a unique solution $x: I \longrightarrow \mathbb{R}^d$ of $\begin{cases} \dot{x}(t) = F(t, x(t)) & \text{for all } t \in \mathbb{R} \\ x(0) = x_0 \end{cases}$

(*). $\dot{x} = F(t,x)$

$$x(t) = x_0 + \int_0^t \dot{x}(s)\,ds = \boxed{x_0 + \int_0^t F(s, x(s))\,ds}$$

integral equation equivalent to the original ODE with $x(0) = x_0$.

<u>Def</u> Picard operator $T(x)$ $T(x)(t) = x_0 + \int_0^t F(s, x(s))\,ds$

$\boxed{x = Tx}$ in other words, solutions of ODE are fixed points of T.

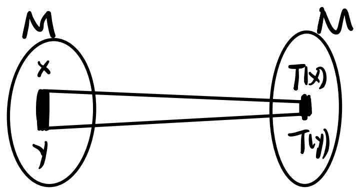
Fixed point theorem:

Def: $x \in M$ is called a <u>fixed point</u> for a map $T: M \to M$ if $T(x) = x$

e.g. $M = [0,1]$ $T(x) = \frac{1}{2}x + \frac{1}{2}$ $T(x) = x \Rightarrow x = 1 \in M$.

Def. A map $T: M \to M$ is called contraction wrt a metric ρ on M if there is a constant $0 < \lambda < 1$ s.t. for all $x, y \in M$

$$\rho(T(x), T(y)) \leq \lambda \rho(x, y)$$



Contraction denotes continuity.

Thm. If T is a contraction on a complete metric space (M, ρ) then there is a unique fix point for T. (unique sol of eqn $x = Tx$)

$$C([a,b], \mathbb{R}^d) = \left\{ \begin{array}{l} \text{continuous functions} \\ x: [a,b] \to \mathbb{R}^d \end{array} \right.$$

$\parallel$
(in Euclidean distance)

Thm. C is complete w.r.t. the sup-metric.

$$\rho(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|$$

Picard method of solving ODEs:

Take any initial approximation x $\quad x \equiv x_0$.

Compute $T(x)$, $T^2(x) = T(T(x))$ $\quad T^n(x) \to y$

We know this converges on $[-\varepsilon, \varepsilon]$ in fact it converges for broader time interval.

Textbook p.35 example.

Flows Smooth Dependence:

$\dot{x}(t) = F(t, x(t))$ (1) in $\mathbb{R}^d$.

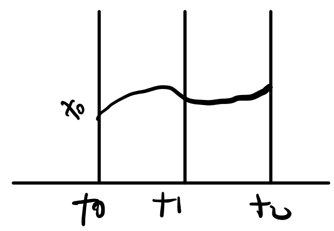
If $F \in \mathrm{Lip}_x$, then for each (t_0, x_0), there's a unique solution $x(t)$ s.t. $x(t_0) = x_0$, $t \in \mathbb{R}$.

Acutually $x(t)$ also depends on x_0, t_0.

Denote t $\Phi(t_0, t, x_0) = \Phi^{t_0 t}(x_0) = \Phi^{t_0 t} x_0$

$\Phi^{t_0 t} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ often called flow solution evolution associated with (1).

Flow property: $\Phi^{t_1 t_2} \Phi^{t_0 t_1} x = \Phi^{t_0 t_2} x$



For autonomous equations: $\dot{x} = F(x)$

Then (HW). $\Phi^{t_0\, t_0+t} x = \Phi^{0 t} x$

$\Phi^{t_0 t_1} = \Phi^{0\, t_1 - t_0}$

introduce: $\Phi^{t} x = \Phi^{0 t} x$

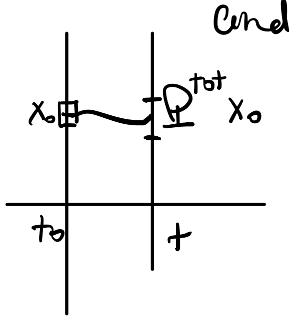
Flow property becomes (group property) $\quad \Phi^{n} = (\Phi^{1})^{n}$

$\Phi^{s} \Phi^{t} x = \Phi^{s+t} x$.

i.e. $\exists C$. $|F(t,x) - F(t,y)| \leq C|x-y|$

THM 1) If F is lipschitz, then $\Phi \in C$.

2) If $F \in C^1$, then Φ is differentiable and satisfies an equation.



Gronwell's lemma:

Suppose for $t \in [0,T]$

$h(t) \le \alpha + C\int_0^t h(s)ds$ (1), $h(t) \ge 0$. $\alpha \ge 0$, $C \ge 0$.

Then $h(t) \le \alpha e^{ct}$, $t \in [0,T]$

solves $\begin{bmatrix} \dot{x} = cx \\ x(0) = \alpha \end{bmatrix}$

Proof. rewrite $\dfrac{h(t)}{\alpha + C\int_0^t h(s)ds} \le 1$

reform $l.h.s = \frac{1}{C}\frac{d}{dt} \ln(\alpha + C\int_0^t h(s)ds \le 1$

take integral $\ln(\alpha + C\int_0^t h(s)ds) - \ln(\alpha + C\int_0^0 \cancel{\ \ }) \le Ct.$

$e^{\square}$ $\dfrac{\alpha + C\int_0^t h(s)ds}{\alpha} \le e^{ct}$.

$\alpha + C\int_0^t h(s)ds \le \alpha e^{ct}$ $\square$

Now we prove (1). continuity: $\Phi^{t_0,t}$

$y(t) = \Phi^{0,t} y_0$ prove $y(t), x(t)$ close if x_0, y_0 close.

$x(t) = \Phi^{0,t} x_0$

$$h(t) := |y(t) - x(t)| = \left| y_0 + \int_0^t F(s, y(s))\,ds - x_0 - \int_0^t F(s, x(s))\,ds \right|$$

$$\leq |x_0 - y_0| + \int_0^t \left| F(s, y(s)) - F(s, x(s)) \right| ds.$$

$$\leq |x_0 - y_0| + \int_0^t C|y(s) - x(s)|\,ds \qquad \text{(Lipschitz)}$$

$$\leq |y_0 - x_0| + C\int_0^t h(s)\,ds$$

Apply Gronwell's lemma:

$$|y(t) - x(t)| \leq |y_0 - x_0| e^{Ct}, \quad e^{CT} \quad t \in [0, T]$$

Remark: same method work for uniqueness.

Thm If $F \in C^1$, then Φ is differentiable.

Find a linearization of the flow Φ^{0t}

$$\Phi^{0t} y = y(t) = y + \int_0^t F(s, y(s))\, ds$$

$$\underbrace{\frac{\partial \Phi^{0t} y}{\partial y}} = 1 + \int_0^t \frac{\partial}{\partial y} F(s, \Phi^{0s} y)\, ds$$

$x = \Phi^{0s} y$

$$= 1 + \int_0^t \frac{\partial}{\partial x} F(s, \Phi^{0s} y) \frac{\partial \Phi^{0s} y}{\partial y}\, ds$$ chain rule.

$$Z(t) = \frac{\partial \Phi^{0t}}{\partial y}. \quad Z(0) = 1. \quad \underset{\frac{\partial z}{\partial t}}{\dot{Z}(t)} = \frac{\partial}{\partial x} F(t, \Phi^{0t} y)\, Z(t)$$

In $\mathbb{R}^d$, $(\Phi^{0t} y) = (\Phi^{0t} y_1, \Phi^{0t} y_2, \ldots \Phi^{0t} y_k)$

$$\frac{\partial \Phi^{0t}}{\partial y} = \left(\frac{\partial \Phi^{0t}}{\partial y_j} \right)_{i,j = 1 \ldots d},$$

$$Z(0) = I.$$

$$\dot{Z}(t) = \frac{\partial F}{\partial x}(t, y(t))\, Z(t)$$

↑ jacobian matrix

$$\frac{\partial F}{\partial x}(t, x) = \left(\frac{\partial F_i}{\partial x_j} \right)_{i,j = 1 \ldots}$$

Euler's method:

$$\frac{x(t+h)-x(t)}{h} \approx \dot{x}(t) \qquad x(t+h)-x(t) = hF(t,x) + o(h)$$

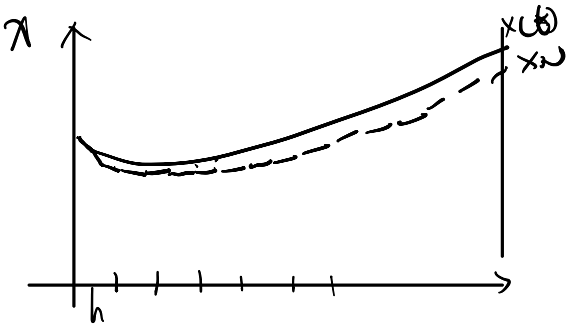
$$\vec{a}_{k+1} = \vec{a}_k + \vec{h} \qquad x_{k+1} = x_k + hF(t_0, x_0).$$

As $h \to 0$, approximates better.

Convergence of Euler

$\dot{x} = F(t,x)$ Assume $F = \mathrm{lip}_x$. $h>0$ small. $h = \frac{t}{N}$

$$\begin{cases} \dot{x} = F(t,x) \\ x(0) = x_0 \end{cases}$$

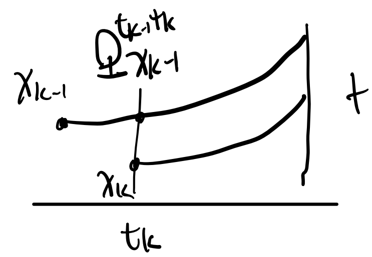


$$\dot{x}(t) = \frac{d}{dt}x(t) = \lim_{r\to 0}\frac{x(t+r)-x(t)}{r}$$

$$x(t+h) = x(t) + \dot{x}(t)h + R(t,h)$$

$$t_k = kh$$

$$x_k = x_{k-1} + F(t_{k-1}, x_{k-1})h$$



$$|x_N - \Phi^{0t}x_0| \leq \sum_{k=1}^{N} |\Phi^{t_k t}x_k - \Phi^{t_{k-1}t}x_{k-1}| \quad (1)$$

$$|\Phi^{t_k t}x_k - \Phi^{t_{k-1}t}x_{k-1}| = |\Phi^{t_k t}x_k - \Phi^{t_k t}\Phi^{t_{k-1}t_k}x_{k-1}|$$

If $|F(t,x) - F(t,y)| \leq C|x-y|$,

$$|\Phi^{st}x_0 - \Phi^{st}y_0| \leq |x_0 - y_0|e^{c(t-s)}$$

$$|\Phi^{0t}x_0 - \Phi^{0t}y_0| \leq |x_0 - y_0|e^{ct}$$

$$\leq |x_k - \Phi^{t_{k-1}, t_k} x_{k-1}| e^{C(t-t_k)}$$

$$\leq |x_k - \Phi^{t_{k-1}, t_k} x_{k-1}| e^{Ct} \quad (2)$$

$$(1)+(2) \longrightarrow |x_N - \Phi^{0,t} x_0| \leq \sum_{k=1}^{N} e^{Ct} |x_k - \Phi^{t_{k-1}, t_k} x_{k-1}|$$

Plug (4) into estimate gives

$$\leq \sum_{k=1}^{N} e^{Ct} \frac{M}{2} h^2 = N e^{Ct} \frac{M}{2} h^2 = t e^{Ct} \frac{M}{2} h$$

?

$$\Phi^{t_{k-1}, t_k} x_{k-1} = x_{k-1} + F(t_{k-1}, x_{k-1}) h + R(t, h) \quad . \quad (\text{linearization})$$

Taylor formula: $R(t,h) = \frac{\ddot{x}(s)}{2} h^2$, $s \in (t, t+h)$

Assume that $|\ddot{x}(t)| \leq M$ for all solutions for some M. (3).

$|R(t,h)| \leq \frac{M}{2} h^2$ (4).

$$\ddot{x}(t) = \frac{d}{dt}(\dot{x}(t))$$

$$= \partial_t F + \partial_x F(t, x(t)) \dot{x}(t)$$

$$= \partial_t F(t, x(t)) + \partial_x F(t, x(t)) F(t, x(t))$$

If assume that $F(t,x)$. $\partial_t F(t,x)$ $\partial_x F(t,x)$

are all well-defined and locally bounded, then assumption (3) holds.

Theorem: If F is Lip_x and $F, \partial_t F, \partial_x F$ are bounded.
then (or we can require $\ddot{x}(t) \leq M$) then
$|x_N - \Phi^{0,t} x_0| \leq K t e^{Ct} h$
In particular, lhs $\to 0$ as $h \to 0$ $(N \to \infty)$
With round off errors: $|\text{error}| \leq \sum e^{Ct} (\frac{M}{2} h^2 + \delta) = K t e^{Ct} + \delta N e^{Ct}$

$$|\text{error}| = \sum e^{ct}\left(\frac{M}{2}h^2 + \delta\right) \leq K t e^{ct} h + \underbrace{N}_{=\frac{t}{h}} e^{ct}\delta$$

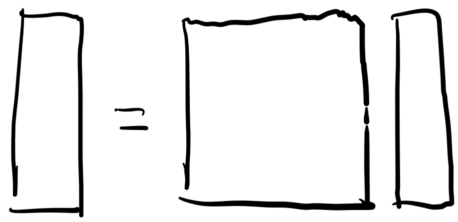
need to choose h that balances the two

Linear autonomous equations:

Solve $\dot{x}(t) = A\, x(t)$. in $\mathbb{R}^d$ (1).

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix}$$

$$\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_d(t) \end{pmatrix}$$



If $d=1$ $\begin{cases} \dot{x} = ax \\ x(0) = x_0 \end{cases}$ $x(t) = e^{at} x_0$.

Thm. Take any $A \in M_d = \{d \times d \text{ matrices}\}$.

$$\begin{cases} \dot{x}(t) = A\,x(t). \\ x(0) = x_0 \end{cases}$$

has a unique solution $x(t) = e^{At} x_0$.

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!} \quad \text{(Taylor ~~the~~ series)}$$
$$= 1 + at + \frac{(at)^2}{2} + \cdots$$

$i = k-1$

$$\frac{d}{dt}(e^{at}) = \sum_{k=1}^{\infty} \frac{a^k}{k!} k t^{k-1} = a \sum_{k=1}^{\infty} \frac{(at)^{k-1}}{(k-1)!} = a \sum_{j=0}^{\infty} \frac{(at)^j}{j!} = a e^{at}$$

$$\frac{d}{dt}(e^{at} x_0) = \cdots = a e^{at} x_0$$

Def. For $A \in M_d$ define.

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2} + \cdots$$

$$x(t) = e^{tA} x_0 = \left(I + A + \frac{A^2}{2} + \cdots\right) x_0$$

$$\dot{x}(t) = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}\right) x_0 = \sum_{k=0}^{\infty} \frac{k t^{k-1} A^k}{k!} x_0$$

order doesn't matter here (?)

$$= A e^{tA} x_0$$

Discuss the convergence of def e^{At}.

M_d is a vector space if $A, B \in M_d$. a, b numero.

$aA + bB \in M_d$.

It's an d squared

Norm of matrix: $\|B\| = \sup_{|x| \le 1} |Bx|$

(review numerical)

$$|Bx| = \left| |x| B \frac{x}{|x|} \right| \le |By|$$

$$y = \frac{x}{|x|},\ |y| = 1$$

$$\rho(A, B) = \|A - B\|$$

Def. $A_n \to A$ in $(M_d, \|\cdot\|)$

if $\|A_n - A\| \to 0$ as $n \to \infty$. if $\sum_{k=1}^{n}$

Def. $\sum_{k=1}^{\infty} A_k = A$ if $\sum_{k=1}^{\infty} A_k \to A$ $n \to \infty$.

Absolute convergence:

For ~~n~~numbers $\sum_{k=1}^{\infty} a_k$ converges absolutely if $\sum_{k=1}^{\infty} |a_k| < \infty$.

Thm: A absolute convergent series converges.

Def: For matrices, if $\sum_{k=1}^{\infty} \|A_k\| < \infty$, then $\sum_{k=1}^{\infty} A_k$ converges absolutely.

Thm: absolute convergent series converges.

Claim 1: e^A is defined by an absolutely convergent series.

Claim 2: $\|AB\| \leq \|A\| \|B\|$ (HW)

$\|A^k\| \leq \|A\| \cdots \|A\| = \|A\|^k$ help prove

Proof of 1: $\sum_{k=0}^{\infty} \left\| \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|} < \infty$

Practical Solving $\dot{x} = Ax$ in $\mathbb{R}^d$.

$x(t) = e^{tA} x(0)$. (1)

$e^{tA} = I + tA + \frac{(tA)^2}{2!} + \cdots$

If A is diagonal, $A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_d \end{pmatrix}$

then computing e^{tA} is easy. $A^k = \begin{pmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_d^k \end{pmatrix}$

$$e^{tA} = \begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_d t} \end{pmatrix}$$

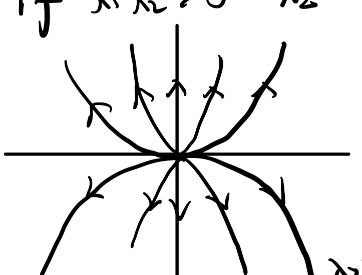
$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 \\ \vdots \\ \dot{x}_d = \lambda_2 x_d \end{cases} (1) \xrightarrow{\text{solves}} \begin{cases} x_1(t) = e^{\lambda_1 t} x_1(0) \\ \vdots \\ x_d(t) = e^{\lambda_1 t} x_d(0). \end{cases}$$

in $\mathbb{R}^2$ $\dot{x}(t) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. $x_1(t) = e^{\lambda_1 t} x_1(0)$ (a) $x_2(t) = e^{\lambda_2 t} x_2(0)$ (b).

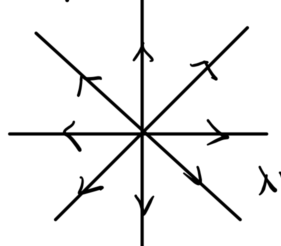
Raise (a) to $\frac{\lambda_2}{\lambda_1}$ power $|x_1(t)|^{\frac{\lambda_2}{\lambda_1}} = e^{\lambda_2 t} |x_1(0)|^{\lambda_2/\lambda_1}$ Combine with (b).

So, $x_2(t) = C|x_1(t)|^{\frac{\lambda_2}{\lambda_1}}$

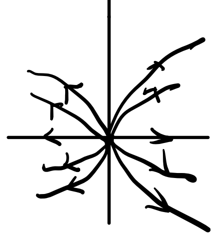
if $\lambda_1, \lambda_2 > 0$ $x_2 = C|x_1|^{\frac{\lambda_2}{\lambda_1}}$



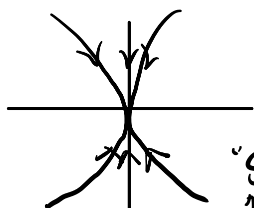
$\lambda_2 > \lambda_1$



$\lambda_1 = \lambda_2$



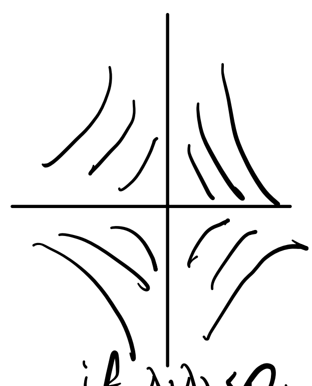
$\lambda_1 > \lambda_2$



$\lambda_1, \lambda_2 < 0$.

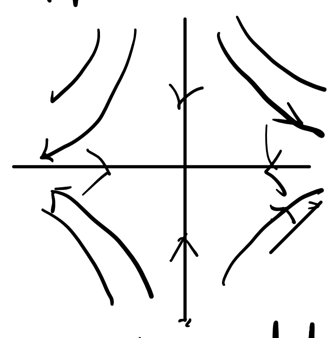
"stable node"

Solutions brought back to 0



if $\lambda_1 \lambda_2 < 0$.

Suppose $\lambda_1 > 0 > \lambda_2$



portrait: "saddle"

Ex: $\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x, \quad x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

① find eigenvalues of A:

$$\det \begin{pmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} = 0, \quad \lambda_1 = -1, \quad \lambda_2 = -2$$

② find eigenvectors of A:

$$(A_i - \lambda_i) v_i = 0$$

$$\begin{bmatrix} -\lambda_i & 1 \\ -2 & -3-\lambda_i \end{bmatrix} v_i = 0$$

$$\Rightarrow v_1 = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_2 = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

③ $$x(t) = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}_{V} \begin{bmatrix} e^{-t} & \\ & e^{-2t} \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}}_{V^{-1}} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{x_0}$$

Ex 2: $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

① Find eigenvalues:

$$\lambda_+ = 2 + i, \quad \lambda_- = 2 - i$$

② Find eigenvectors:

$$v_+ = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_1 + i v_2$$

$$v_- = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_1 - i v_2$$

New basis (v_1, v_2)

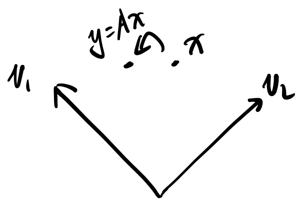
$$A v_+ = \lambda_+ v_+ = (2+i)(v_1 + i v_2)$$
$$\| $$
$$A(v_1 + i v_2)$$
$$\|$$
$$A v_1 + i A v_2 = 2v_1 - v_2 + i(v_1 + 2v_2)$$

So $A v_1 = 2v_1 - v_2$

$A v_2 = v_1 + 2 v_2$

In the basis (v_1, v_2), the linear map is described by $D = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

Change of Coordinates



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x = c_1 v_1 + c_2 v_2$$
$$= c_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} + c_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$
$$= \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y = d_1 v_1 + d_2 v_2$$
$$= \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

Goal: Express d in terms of c.

$V = \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix}$, then $x = Vc$, $y = Vd$, $y = Ax$.

Then we may write $d = V^{-1} y = V^{-1} A x = \underbrace{V^{-1} A V}_{\text{similar to } A} c$ $\quad d = Dc$.

So the transformation in new coordinates is given by $D = V^{-1} A V$

Last time: If v_1, v_2 are noncolinear eigenvectors, then

$D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$. λ_1, λ_2 are eigenvalues and

$$A v_1 = \lambda_1 v_1, \quad A v_2 = \lambda_2 v_2$$

$$A = V D V^{-1}, \quad e^{tA} = V e^{tD} V^{-1} = V \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} V^{-1}.$$

To compute e^D for $D = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$

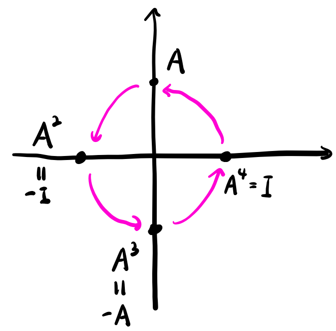
For $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $e^{tA} = I + tA + \frac{t^2A^2}{2!} + \cdots$

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$A^3 = -IA = -A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^4 = (-I)(-I) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A^0$$

$$A^5 = A^4 A = IA = A$$



So
$$e^{tA} = \begin{pmatrix} \underbrace{1 - \frac{t^2}{2} + \frac{t^4}{4!} + \cdots + \frac{(-1)^{n+1} t^{2n}}{(2n)!}}_{\cos t} & -\sin t \\ \underbrace{t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots + \frac{(-1)^{n+1} t^{2n+1}}{(2n+1)!}}_{\sin t} & \cos t \end{pmatrix}$$

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases}$$

$$e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$ rotation with angular velocity 1.

$$\Rightarrow e^{t\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}} = \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}$$ rotation with angular velocity β.

$$\Rightarrow e^{t\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}} \overset{(*)}{=} e^{t\alpha I} e^{t\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}} = \begin{pmatrix} e^{t\alpha} & 0 \\ 0 & e^{t\alpha} \end{pmatrix}\begin{pmatrix} \cos t\beta & -\sin t\beta \\ \sin t\beta & \cos t\beta \end{pmatrix}$$

$$= \begin{pmatrix} e^{t\alpha}\cos(t\beta) & -e^{t\alpha}\sin(t\beta) \\ e^{t\alpha}\sin(t\beta) & e^{t\alpha}\cos(t\beta) \end{pmatrix}$$

(*) **Thm.** If A and B commute, then $e^{A+B} = e^A e^B = e^B e^A$.

Let us consider a generic A.

Suppose we know $Av = \lambda v$ (λ e. value $/\mathbb{R}$, v e. vector $\mathbb{R}^d$).

Then, $x(t) = Ce^{\lambda t} v$ is a solution of $\dot{x} = Ax$,)

$$\dot{x}(t) = C\lambda e^{\lambda t} v = A[Ce^{\lambda t} v] = Ax(t) \text{ (verify).}$$

Let $A \in M_2$, $\lambda_1 v_1$ eigenpair, $\lambda_2 v_2$ 〃—〃.

represent $x = c_1 v_1 + c_2 v_2$ (2).

$x(t) = C$

$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ (3).

$\dot{x}(t) = A c_1 e^{\lambda_1 t} v_1 + A c_2 e^{\lambda_2 t} v_2$

$= A(c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2) = Ax(t)$.

$\dot{x} = Ax$

$\lambda_2, \lambda_1 \in \mathbb{R}$, $v_2 \neq 0$, $v_1 \neq 0$

$\lambda_1 \lambda_2$ form basis.

1) find e.v. pairs.

2) represent $x = \cdots$ (find c_1 c_2)

3) write (3).

$\lambda_1, \lambda_2 > 0$ unstable node

$\lambda_1, \lambda_2 < 0$ stable node.

Diagonalize:

$$A = VDV^{-1} \rightarrow e^{tA} = Ve^{tD}V^{-1}$$

Rewrite (2).

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} + c_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$= \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = V \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$Ax = c_1 A v_1 + c_2 A v_2$$

$$= c_1 \lambda_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} + c_2 \lambda_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$= V \begin{pmatrix} c_1 \lambda_1 \\ c_2 \lambda_2 \end{pmatrix}$$

$$= V \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

recall

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = V^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$Ax = V \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V^{-1} x$$

<u>Thm</u>. If v_1, λ_1, v_2, λ_2 are e.v. pairs for A s.t. v_1, v_2 are linearly independent. then

$$x(t) = \underbrace{V \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} V^{-1}}_{e^{At}} x(0).$$

$Av_1 = 2v_1 - v_2$ In the new basis (v_1, v_2), the

$Av_2 = v_1 + 2v_2$ linear map is expressed by $D = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

New goal is to compute e^D where $D = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

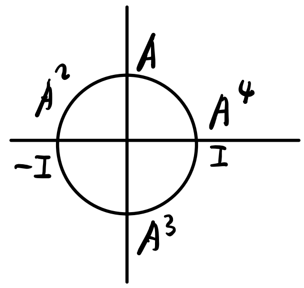
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

$$A^3 = A^2 A = -A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^4 = A^2 A^2 = I = A^0.$$

$\cdots$



$$e^{tA} = I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \frac{t^4A^4}{4!}$$

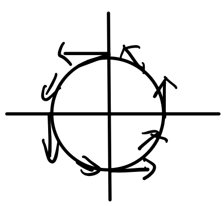
$$= \begin{pmatrix} 1 - \frac{t^2}{2} + \frac{t^4}{4!} + \cdots + \frac{(-1)^n t^{2n}}{(2n)!} & --- \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots + \frac{(-1)^n t^{2n+1}}{(2n+1)!} & --- \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Hence, $e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. rotation.

$\dot{x}_1 = -x_2$

$\dot{x}_2 = x_1$



$$e^{t\begin{pmatrix}0 & -\beta\\ \beta & 0\end{pmatrix}} = e^{t\beta\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}} = \begin{pmatrix}\cos\beta t & -\sin\beta t\\ \sin\beta t & \cos\beta t\end{pmatrix}.$$

the general matrix $D=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix} = \alpha I + \begin{pmatrix}0 & -\beta\\ \beta & 0\end{pmatrix}.$

$$e^{tD} = e^{t\alpha I}\, e^{t\begin{pmatrix}0 & -\beta\\ \beta & 0\end{pmatrix}}$$

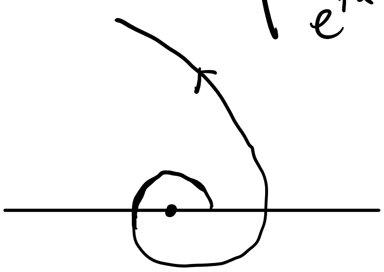
Thm. If A and B commute $AB=BA$. then $e^{A+B} = e^{A}e^{B} = e^{B}e^{A}$.

Because $t\alpha I$ commutes with every other matrix

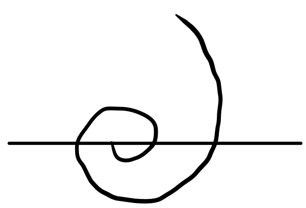
$$e^{tD} = e^{t\alpha I}\, e^{t\begin{pmatrix}0 & \beta\\ -\beta & 0\end{pmatrix}}$$

$$= \begin{pmatrix}e^{t\alpha} & 0\\ 0 & e^{t\alpha}\end{pmatrix}\begin{pmatrix}\cos t\beta & -\sin t\beta\\ \sin t\beta & \cos t\beta\end{pmatrix}$$

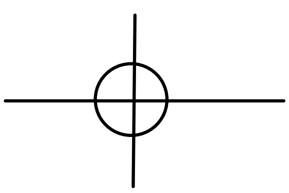
$$= \begin{pmatrix}e^{t\alpha}\cos t\beta & -e^{t\alpha}\sin t\beta\\ e^{t\alpha}\sin t\beta & e^{t\alpha}\cos t\beta\end{pmatrix}$$



$\alpha > 0.$



$\alpha < 0$



$\alpha = 0$

$\dot{x} = Ax$ in $\mathbb{R}^n$.

$x(0) = x^0$

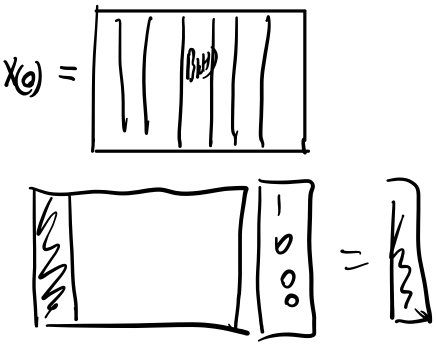
$x^0 = (x_1^0, x_2^0, \ldots, x_n^0)$.

$x(t) = e^{tA}x(0)$.

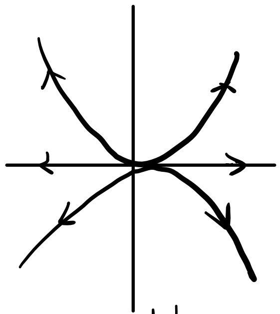
e^{tA} is called the fundamental solution.

$$B^k(t) = e^{tA}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = e^{tA}e_k.$$

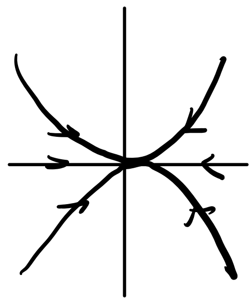
$e^{tA}x(0) =$



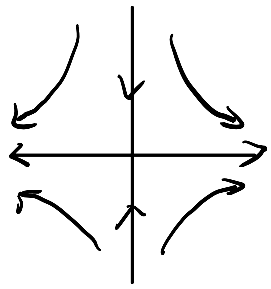
Node behavior:



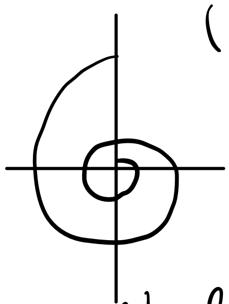
unstable



stable



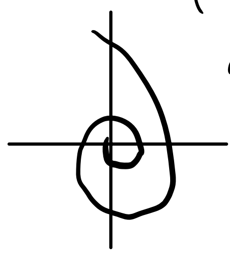
~~sta~~ saddle.



$\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$

$\alpha > 0$.

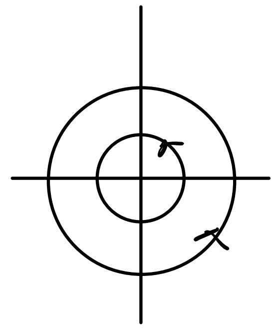
stable focus



$\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$

$\alpha < 0$

unstable focus



$\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$

$\alpha = 0$,

$0 \pm i\beta$.

$$A = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} \qquad f(x) = (\lambda - x)^2.$$

$$A = \begin{pmatrix}\lambda & 1 & & & \\ & \lambda & 1 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & \lambda\end{pmatrix} \leftarrow \text{Jordan Cell.}$$

$$= \begin{pmatrix}\lambda & & & 0\\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda\end{pmatrix} + \underbrace{\begin{pmatrix}0 & 1 & & & 0\\ & 0 & 1 & & \\ & & 0 & 1 & \\ & 0 & & 0 & 1\\ & & & & 0\end{pmatrix}}_{B} = \lambda I + B.$$

$$e^{tA} = e^{t(\lambda I + B)} = e^{t\lambda I} e^{tB} = e^{t\lambda} e^{tB}.$$

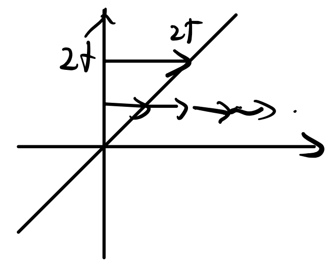
$$B^2 = \begin{pmatrix}0 & 0 & 1 & & & 0\\ & 0 & 0 & 1 & & \\ & & 0 & 0 & 1 & \\ & 0 & & 0 & 0 & 1\\ & & & & 0 & 0 \\ & & & & & 0\end{pmatrix} \qquad B^{n-1} = \begin{pmatrix} & & 0 & 1\\ & 0 & & 0 \end{pmatrix}. \quad B^n = (0).$$

$$e^{tB} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \frac{t^2}{2} & \\ & & 1 & t & \frac{t^2}{2} \\ & & & \ddots & \frac{t^2}{2} \\ & & & & t \\ & & & & 1 \end{pmatrix}$$

"Sheer Flow"

If B is 2×2. $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}$.



X axis stays same.

$D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. and $D = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

$\underset{2\times 2}{A}$. $\chi(A) = \det(A - \lambda I)$

Just one root λ

$D - \lambda I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. any vector is solution to $(D-\lambda I)v = 0$.

$D - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ The e-vector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_2$

$(D-\lambda I)v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_1$.

v_1, v_2 form the canonical basis.

E.X. $A=\begin{pmatrix}1&1\\-1&3\end{pmatrix}$. $\lambda=2$. $v_1=\begin{pmatrix}1\\1\end{pmatrix}$.

$(A-\lambda I)v_2 = v_1$ $\quad v_2=\begin{pmatrix}1\\2\end{pmatrix}$.

(v_1, v_2) will be the new basis.

$V=\begin{pmatrix}1&1\\1&2\end{pmatrix}$. $\quad D=\begin{pmatrix}2&1\\0&2\end{pmatrix}$ $\quad 2\times 2$ Jordan Cell with $\lambda=2$.

$A=VDV^{-1}$ $\quad V^{-1}=\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$.

Compare with e. vector decomps

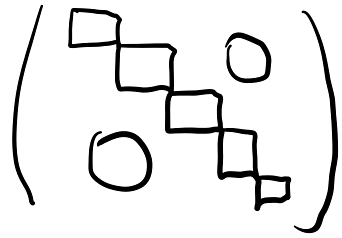
$$A=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}2&1\\0&2\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix}.$$

$$e^{tA}=Ve^{tD}V^{-1}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}e^{2t}&e^{2t}t\\0&e^{2t}\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix}.$$

Solving linear ODEs in high dimensions in general.

Jordan Normal Form:

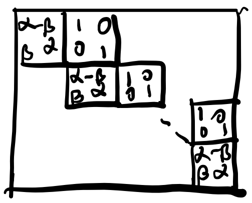
For any $n \times n$ matrix A, there is a basis (Jordan) s.t. in that matrix the operator $x \mapsto Ax$ is a new matrix



Each cell is $\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \\ & & & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

"best is when of size 1 $\boxed{\lambda}$.

or



for some $\alpha, \beta \in \mathbb{R}$.

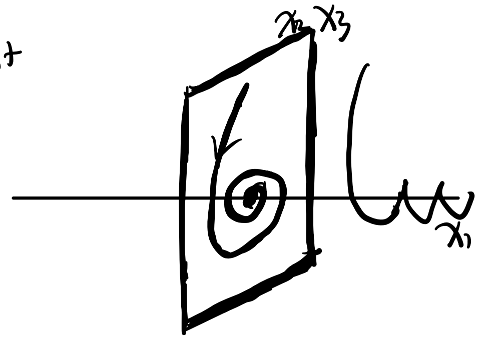
$\lambda = \alpha + i\beta$.

Solutions always have terms:

$e^{\lambda t}$, $te^{\lambda t}$, $t^2 e^{\lambda t}$, ...

$e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$.

$t^k e^{\alpha t}\cos\beta t$. $t^k e^{\alpha t}\cos\beta t$



JNF: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 3 & -2 \end{pmatrix}$

$$\exp(tA) = \exp\left(t\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 3 & -2 \end{pmatrix}\right) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-2t}\cos 3t & -e^{-2t}\sin 3t \\ 0 & e^{-2t}\sin 3t & e^{-2t}\cos 3t \end{pmatrix}$$

Non homogeneous equations:

$\dot{x} = Ax + F(t)$. in $\mathbb{R}_n$.

Variation of constants.

$\dot{x} = Ax \quad (H)$.

$x(t) = e^{tA} x(0)$

Try to find solution of the (N) $x(t) = e^{tA} C(t)$.

$\frac{d}{dt}(A(t)B(t)) = \dot{A}(t)B(t) + A(t)\dot{B}(t)$

$$C(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix}$$

$$\boxed{e^{-tA} e^{tA} = I}$$

$Ae^{tA}C(t) + e^{tA}\dot{C}(t) = Ae^{tA}C(t) + F(t)$.

$\dot{C}(t) = e^{-tA}F(t)$.

$C(t) = C(0) + \int_0^t e^{-sA} F(s)\, ds$.

$\therefore x(t) = e^{tA}\left(C(0) + \int_0^t e^{-sA} F(s)\, ds\right)$.

$x(0) = C(0)$.

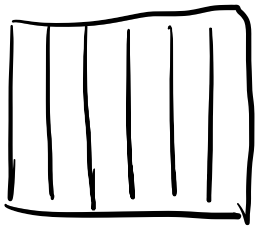
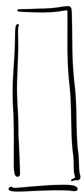
$$\boxed{x(t) = e^{tA}\left(x(0) + \int_0^t e^{-sA} F(s)\, ds\right).}$$

$$= \underbrace{e^{tA} x(0)}_{\text{sol of (H)}} + \underbrace{\int_0^t e^{(t-s)A} F(s)\, ds}_{\text{sol of (N)}}$$

x, y solve (H) $\quad z, w$ solve (N).

$ax + by$ solve (H). $\quad x + z$ solve (N) $\quad z - w$ solve (H)

There are n independent solutions of (N).

$e^{tA} =$  $\quad e^{tA} e_k =$ 

Linear equation of high order:

$$\frac{dx}{dt} = \dot{x}(t). \quad \frac{d^2}{dt^2} x = \ddot{x}(t) \quad \cdots \quad \frac{d^n}{dt^n} x(t) = x^{(n)} t$$

$$x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0 . \quad (1)$$

Change of variables:

$$y(t) \in \mathbb{R}^n \quad y_1 = x \quad y_2 = \dot{x} \quad \cdots \quad y_n = x^{(n-1)}$$

$$\dot{y}_1 = \dot{x} = y_2 . \quad \dot{y}_2 = \ddot{x} = y_3 \quad \cdots,$$

$$\dot{y}_{n-1} = \dot{x}^{(n-2)} = x^{(n-1)} = y_n$$

$$\dot{y}_n = \dot{x}^{(n-1)} = x^{(n)} = -a_0 y_1 - a_1 y_2 + \cdots + a_{n-1} y_n .$$

$$\dot{y} = Ay = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_0 & -a_1 & \cdots & \end{pmatrix} y$$

Need to assign $y(0) = \begin{pmatrix} y_1(0) \\ \vdots \\ y_n(0) \end{pmatrix} = \begin{pmatrix} x(0) \\ \dot{x}(0) \\ \vdots \\ x^{(n-1)}(0) \end{pmatrix}$

$y(t) = e^{tA} \begin{pmatrix} x(0) \\ \vdots \\ x^{(n-1)}(0) \end{pmatrix}$ interested in $x(t) = y_1(t)$.

Solutions always look like $t^k e^{\lambda t}$, $t^k e^{\alpha t} \cos(\beta t)$.
$t^k e^{\alpha t} \sin(\beta t)$

Try plugging $e^{\lambda t}$ into (1)

$$\frac{d^k}{dt^k} e^{\lambda t} = \lambda^k e^{\lambda t}$$

$$\lambda^n e^{\lambda t} + a_{n-1}\lambda^{n-1} e^{\lambda t} + a_{n-2}\lambda^{n-2} e^{\lambda t} + \cdots + a_2\lambda^2 e^{\lambda t} + a_1\lambda e^{\lambda t} + a_0 e^{\lambda t} = 0$$

$$\Rightarrow \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

solvable to λ suggests whether there is solution

The characteristic equation. The l.h.s $\chi(\lambda)$ is char. polynomial.

Suppose $\lambda_1 \cdots \lambda_n$ are distinct solutions of χ,

The equation solution is $x(t) = c_1 e^{\lambda_1 t} + \cdots + c_n e^{\lambda_n t}$.

Ex. $\ddot{x} - 3\dot{x} + 2x = 0 \quad x(0) = 1 \quad \dot{x}(0) = 3$

$\chi = \lambda^2 - 3\lambda + 2, \quad \lambda_1 = 1, \lambda_2 = 2.$

$x(t) = c_1 e^t + c_2 e^{2t}$

Plug in $t=0$

$x(0) = c_2 + c_1 = 1.$

$\dot{x}(t) = c_1 e^t + 2c_2 e^{2t}$

$\dot{x}(0) = c_1 + 2c_2 = 3.$

$\Rightarrow c_1 = -1, \ c_2 = 2.$

$x(t) = -e^t + 2e^{2t}$

$x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 \dot{x} + a_0 x = 0.$ (1)

$t^k e^{\lambda t} \cos(\beta t)$ Try $e^{\lambda t}$ $f(\lambda) = 0.$

$\sin(\beta t).$

$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$

What if f have complex roots?

$\lambda_\pm = \alpha \pm i\beta$

$e^{\lambda_\pm t}$ are solutions. complex.

For real valued solutions

$e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta).$

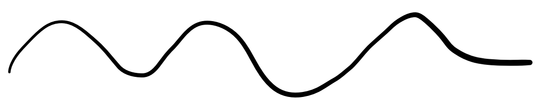
$\frac{1}{2}(e^{\lambda_+ t} + e^{\lambda_- t}) = e^{\alpha t}\cos(\beta t).$

$\frac{1}{2i}(e^{\lambda_+ t} - e^{\lambda_- t}) = e^{\alpha t}\sin(\beta t).$

$\Big\}$ Take linear combination of these solutions.

Applications:

Harmonic Oscillator



$m\ddot{x} = F(x)$

for the spring, $F(x) = -kx.$

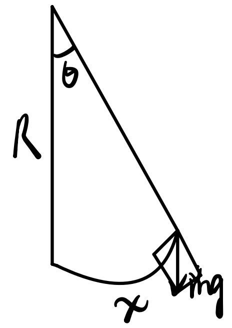
$\ddot{x} + \frac{k}{m}x = 0.$ $n=2,\ a_1 = 0,\ a_0 = \frac{k}{m}.$ $f(\lambda) = \lambda^2 + \frac{k}{m}$ $\lambda = \pm i\sqrt{\frac{k}{m}}.$

$\omega_0 = \sqrt{\frac{k}{m}}$. $\lambda_\pm = \pm i\omega_0 = 0 \pm i\omega_0.$

Solutions: $c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$

$\omega_0 T = 2\pi,\ T = \frac{2\pi}{\omega_0}$ (T is period). $f = \frac{1}{T}.$

Pendulum: $x =$ (signed) length of the displacement arc.



$x = R\theta$.

$m\ddot{x} = -mg\sin\theta$.

$\ddot{x} = R\ddot{\theta}$

$R\ddot{\theta} = -g\sin\theta$

$\ddot{\theta} + \frac{g}{R}\sin\theta = 0$.

If θ small, $\ddot{\theta} + \frac{g}{R}\theta = 0$ $\quad \lambda \pm i\sqrt{\frac{g}{R}}$

$\Rightarrow x = c_1\cos\left(\sqrt{\frac{g}{R}}t\right) + c_2\sin\left(\sqrt{\frac{g}{R}}t\right)$. Period $T = 2\pi\sqrt{\frac{R}{g}}$.

Real Case:

1) Damped oscillation: $F(x) = -kx - \overset{\text{friction}}{u\dot{x}}$ $\quad \delta < w_0$

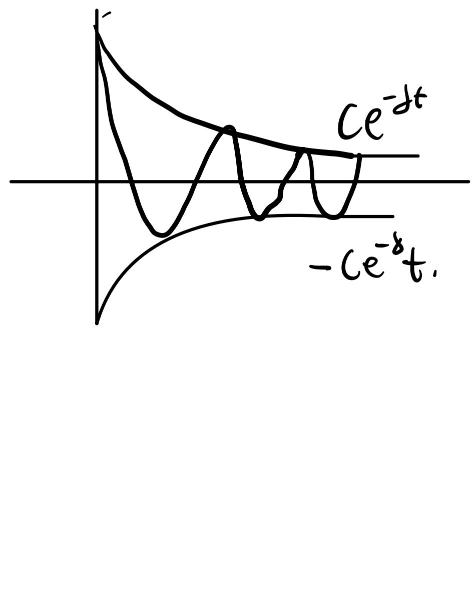
$m\ddot{x} = -kx - u\dot{x}$ $\qquad \ddot{x} + \frac{u}{m}\dot{x} + kx = 0$

$m\ddot{x} + u\dot{x} + kx = 0$, $\quad$ Substitute into (1).

$\lambda = -\frac{u}{2m} \pm \sqrt{\left(\frac{u}{2m}\right)^2 - \frac{k}{m}} = -\delta \pm \sqrt{\delta^2 - w_0^2} = -\delta \pm i\sqrt{w_0^2 - \delta^2}$

1) understood oscillations $\delta < w_0$. (small friction)

Solutions: $c_1 e^{-\delta t}\cos\left(\sqrt{w_0^2-\delta^2}\,t\right) + c_2 e^{-\delta t}\sin\left(\sqrt{w_0^2-\delta^2}\,t\right)$.



2) overdamped oscillation: $\delta > w_0$

$\lambda_+ = -\delta + \sqrt{\delta^2 - w_0^2}$ $\qquad \lambda_- = -\delta - \sqrt{\delta^2 - w_0^2}$

Solutions: $c_1 e^{(-\delta + \sqrt{\delta^2 - w_0^2})t} + c_2 e^{(-\delta - \sqrt{\delta^2 - w_0^2})t}$

oscillate at most once! Friction too large

3) Critical damping $\delta = w_0$

$\lambda = -\delta$, $e^{-\delta t}$ is a solution, $te^{-\delta t}$ is also solution (Jordan).

General: $x(t) = c_1 e^{-\delta t} + c_2 t e^{-\delta t}$

Fact: For a homogeneous autonomous linear ODE, if λ is real root of a characteristic polynomial of multiplicity $K \in \mathbb{N}$.

Then $e^{\lambda t}$, $te^{\lambda t}$, $t^2e^{\lambda t}$, ---, $t^{k-1}e^{\lambda t}$ are solutions

If $\lambda_+ = \alpha + i\beta$. $\lambda_- = \alpha - i\beta$. is pair of conjugate roots, of multi K.

Then $e^{\alpha t}\cos\beta t$, $te^{\alpha t}\cos\beta t$ --- $t^{k-1}e^{\alpha t}\cos\beta t$

$e^{\alpha t}\sin\beta t$ $te^{\alpha t}\sin\beta t$ --- $t^{k-1}e^{\alpha t}\sin\beta t$.

Non-homogeneous 2nd order linear equations: (constant coefficients).

$\ddot{x} + p\dot{x}(t) + q x(t) = f(t)$ (N).

We already know $\ddot{x} + p\dot{x}(t) + q x(t) = 0$. has 2 linearly independent solutions. $y_1(t)$. $y_2(t)$ general $x_H(t) = c_1 y_1(t) + c_2 y_2(t)$

Still Variation of constants,

$\dot{x}(t) = \dot{c}_1 y_1 + c_1 \dot{y}_1 + \dot{c}_2 y_2 + c_2 \dot{y}_2$

$\ddot{x}(t) = (\dot{c}_1 y_1 + \dot{c}_2 y_2)^{\cdot} + \dot{c}_1\dot{y}_1 + c_1\ddot{y}_1 + \dot{c}_2\dot{y}_2 + c_2\ddot{y}_2$

Plug into (N).

$c_1\underbrace{(\ddot{y}_1 + p\dot{y}_1 + q y_1)}_{=0} + c_2\underbrace{(\ddot{y}_2 + p\dot{y}_2 + q y_2)}_{=0} + \underbrace{(\dot{c}_1 y_1 + \dot{c}_2 y_2)^{\cdot}}_{(\text{IV})} +$

(III) $p(\dot{c}_1 y_1 + \dot{c}_2 y_2) + (\dot{c}_1 \dot{y}_1 + \dot{c}_2 \dot{y}_2) = f(t)$.

Just looking for one solution, III, IV $= 0$ for all times

Then, we obtain $\dot{c}_1\dot{y}_1 + \dot{c}_2\dot{y}_2 = f(t)$

$$\begin{cases} \dot{c}_1(t)y_1(t) + \dot{c}_2(t)y_2(t) = 0 \\ \dot{c}_1(t)\dot{y}_1(t) + \dot{c}_2(t)\dot{y}_2(t) = f(t). \end{cases}$$

2×2 linear system eq.

find $c_1 = \int \dot{c}_1 dt + A_1 \quad c_2 = \int \dot{c}_2 dt + A_2$

$$\begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix} \begin{pmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Cramer's Rule:

$$D = \det \begin{pmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{pmatrix}, \quad D_1 = \det \begin{pmatrix} 0 & y_2 \\ f(t) & \dot{y}_2(t) \end{pmatrix}$$

$$D_2 = \det \begin{pmatrix} y_1 & 0 \\ \dot{y}_1 & f(t) \end{pmatrix}$$

$$\dot{c}_1(t) = \frac{D_1}{D} = \frac{-y_2(t) f(t)}{y_1(t)\dot{y}_2(t) - \dot{y}_1(t) y_2(t)}$$

$$\dot{c}_2(t) = \frac{D_2}{D} = \frac{y_1(t) f(t)}{y_1(t)\dot{y}_2(t) - \dot{y}_1(t) y_2(t)}$$

$$c_1(t) y_1(t) + c_2(t) y_2(t) + \underbrace{A_1 y_1(t) + A_2 y_2(t)}_{\text{const.}}$$

EX $\ddot{x} + x = \sin(2t)$, $y_1(t) = \cos t$. $y_2(t) = \sin t$.

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}. \quad . \; D\cdot$$

$\dot{c}_1 = -\sin t \sin 2t. \quad \dot{c}_2 = \cos t \sin 2t.$

$$\int \dot{c}_1 dt = -2\int \sin t \sin t \cos t \, dt$$
$$= -2 \int \sin^2 t \, d\sin t$$
$$= -\frac{2}{3} \sin^3 t + A_1$$

$\int c_2 \, dt = \cdots = \frac{2}{3}\cos^3 t + A_2$.

$x_N(t) = \left(-\frac{2}{3}\sin^3 t + A_1\right)\cos t + \left(-\frac{2}{3}\cos^3 t + A_2\right)\sin t$.

Situations where one can guess solutions:

$\ddot{x} + x = t$, $\quad \ddot{x} + x = t^2$. $\Rightarrow$ generally $f(t) = P_n(t)$

Guess $x = t$. $\quad$ Set $x(t) = at^2 + bt + c$. $\quad x(t) = Q_n(t)$

$\ddot{x} + p\dot{x} + qx = e^{\lambda t}$. Try $x(t) = Ce^{\lambda t}$.

Plug $\rightarrow$ $C(\lambda^2 + p\lambda + q\lambda) = 1$.

$$C = \frac{1}{f(\lambda)}$$

Ex) $\ddot{x} + x = e^{-5t}$. $f(\lambda) = \lambda^2 + 1 = 26$ $\quad x(t) = \frac{e^{-5t}}{26} + A_1\cos t + A_2 \sin t$

This works if $\lambda = \alpha + i\beta$. $\ddot{x} + p\dot{x} + qx = e^{\alpha t}(A_1\cos\beta t + A_2\sin\beta t)$ $\quad (*)$

$$= A_1' e^{(\alpha+i\beta)t} + A_2' e^{(\alpha+i\beta)t}$$

If $f(\lambda) = f(\alpha + i\beta) \neq 0$

$B_1' e^{(\alpha+i\beta)t}$ is a solution of $\ddot{x} + p\dot{x} + qx = A_1' e^{(\alpha+i\beta)t} = A_1' e^{\lambda t}$.

if B_1' is chosen appropriately. $B_1' = \frac{A_1'}{f(\alpha+i\beta)}$

$B_2' e^{(\alpha-i\beta)t}$ is a sol of $\ddot{x} + p\dot{x} + qx = A_2' e^{(\alpha-i\beta)t} = A_2' e^{\bar{\lambda} t}$

where $B_2' = \frac{A_2'}{f(\alpha-i\beta)}$

So, $B_1' e^{(\alpha+i\beta)t} + B_2' e^{(\alpha-i\beta)t}$ solves $(*)$ Real valued function

$(\ldots)$.

Continue (...).

Rewrite as $B_1 e^{\alpha t}\cos(\beta t) + B_2 e^{\alpha t}\sin(\beta t)$, $B_1, B_2 \in \mathbb{R}$.

We can set $x(t) = \uparrow$ and find B_1, B_2

Ex] $\ddot{x} + x = e^t \sin t$.

$x(t) = B_1 e^t \cos t + B_2 e^t \sin t$.

$\dot{x}(t) = B_1(e^t\cos t - e^t \sin t) + B_2(e^t \sin t + e^t \cos t)$

$= (B_1 + B_2) e^t \cos t + (B_2 - B_1) e^t \sin t$,

$\ddot{x}(t) = 2B_2 e^t \cos t - 2B_1 e^t \sin t$

Plug in, $\underbrace{(2B_2 + B_1)}_{=0}\cos t + (\underbrace{-2B_1 + B_2}_{=0} - 1)\sin t = 0$

$B_1 = -\frac{2}{5}$ $B_1 = \frac{1}{5}$.

$\ddot{x} - x = e^t$. Try $x(t) = Ce^t$, $\lambda \pm = \pm 1$

instead try $\begin{cases} x(t) = cte^t \\ \dot{x}(t) = c(e^t + te^t) \\ \ddot{x} = c(2e^t + te^t) \end{cases}$ $\xrightarrow{\text{plug}}$ $x(t) = \frac{1}{2}te^t$ is a sol.

general sol: $\frac{1}{2}te^t + c_1 e^t + c_2 e^{-t}$.

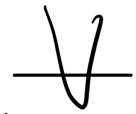
$\ddot{x} + p\dot{x} + qx = e^{\lambda t}$. $\chi(\lambda) = 0$

Try $x(t) = cte^{\lambda t}$.

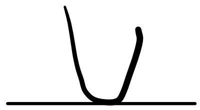
$\dot{x}(t) = c(e^{\lambda t} + \lambda t e^{\lambda t})$

$\ddot{x}(t) = c(\lambda e^{\lambda t} + \lambda e^{\lambda t} + \lambda^2 t e^{\lambda t}) = c(2\lambda e^{\lambda t} + \lambda^2 t e^{\lambda t})$

Plug in -- $c(t\underbrace{(\lambda^2 + p\lambda + q)}_{=0} + 2\lambda + p) = 1$.

$C(2\lambda+p)=1$, $C=\frac{1}{2\lambda+p}$ $(2\lambda+p\neq 0)$ 

$\frac{d}{d\lambda}f(\lambda)=2\lambda+p$ ↓ derivative not 0. → λ simple root

But, if $2\lambda+p=0$.  then try $x(t)=Ct^2e^{\lambda t}$

$\ddot{x}+w_0^2x=\cos\beta t$

homog eq. ($\ddot{x}+w_0^2x=0$) has sol $C_1\cos(w_0t)+C_2\sin(w_0t)$.

Guess for a sol for non homog.

$x(t)=a_1\cos\beta t+a_2\sin(\beta t)$

$\ddot{x}=-a_1\beta^2\cos(\beta t)-a_2\beta^2\sin(\beta t)$.

Plug in. if $\beta^2\neq w_0^2$ $a_1(w_0^2-\beta^2)=1$. $a_2(w_0^2-\beta^2)=0$

$a_1=\frac{1}{w_0^2-\beta^2}$ $a_2=0$.

general: $x(t)=\frac{1}{w_0^2-\beta^2}\cos(\beta t)+C_1\cos(w_0t)+C_2\sin(w_0t)$.

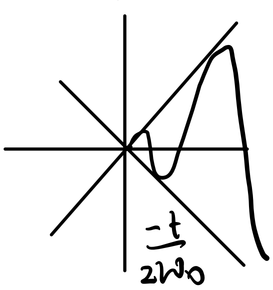
If $\beta=w_0$.

general $x(t)=a_1t\cos(w_0t)+a_2t\sin(w_0t)$.

$-2a_1w_0\sin(w_0t)+2a_2w_0\cos(w_0t)=\cos(w_0t)$.

$a_1=0$. $a_2=\frac{1}{2w_0}$

So, $x(t)=\frac{1}{2w_0}t\sin(w_0t)+C_1\cos(w_0t)+C_2\sin(w_0t)$. is general sol.



Resonance

Non-linear multidimension system

Hamitonian Systems

Turns out that every system can be written as follows:

phase space: $\mathbb{R}^2 \to (q, p)$

coordinate momentum.

$H: \mathbb{R}^2 \to \mathbb{R}$ "energy function" or "Hamiltonian"

Hamiltonian ODE with Hamiltonian H:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p}(q, p). \\ \dot{p} = -\frac{\partial H}{\partial q}(q, p) \end{cases}$$

Classical Example:

$H(q, p) = \frac{p^2}{2m} + V(q)$ $\quad$ q: position of the particle

$p = m\dot{q}$: momentum of the particle $\quad v = \dot{q}$

$\frac{p^2}{2m} = \frac{(m\dot{q})^2}{2m} = \frac{1}{2}m(\dot{q})^2 = \frac{1}{2}mv^2$ kinetic energy

$V(q)$: potential energy

What is potential energy?

Force F, $\mathbb{R} \to \mathbb{R}$. $F(q)$ = force acting on particle located at q.

We can always find $V: \mathbb{R} \to \mathbb{R}$ s.t. $-\frac{dV}{dq}(q) = F(q)$.

$$V = -\int F(q)\,dq$$

$H(q, p) = \frac{p^2}{2m} + V(q)$

$$\begin{cases} \dot{q} = \frac{\partial H(q,p)}{\partial p} \\ \dot{p} = -\frac{\partial H(q,p)}{\partial q} \end{cases}$$

check $\dot{q} = \frac{2p}{2m} = \frac{m\dot{q}}{m} = \dot{q}$ (v)

$\dot{p} = -\frac{\partial V(q)}{\partial p} = F(q)$ (v) 牛=

Conservation of energy:

$$\frac{d}{dt} H(q(t), p(t)) = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}$$

$$= \frac{\partial H}{\partial q}\cdot\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\cdot\left(-\frac{\partial H}{\partial q}\right) = 0.$$

$H(q(t), p(t)) = \text{const.}$

Def: We say I is preserved by the dynamics invariant under the flow

For the classical system:

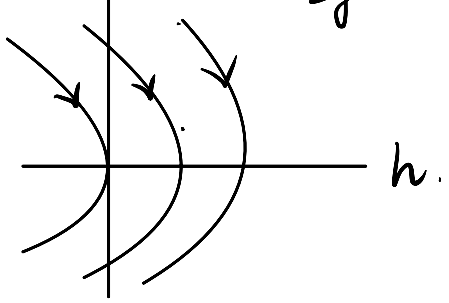
$$\frac{p^2}{2m} + V(q) = \text{const.}$$

Gravity: $V(h) = mgh.$

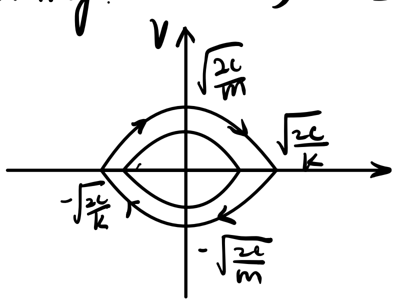
$$\frac{m\dot{h}^2}{2} + mgh = C \qquad \dot{h} = v.$$

$$\frac{1}{2}mv^2 + mgh = C$$

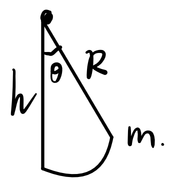
$V = \dot{h}$ $\quad h = -\frac{v^2}{2g} + C'$



Spring: $E(v, x) = \frac{mv^2}{2} + \frac{kx^2}{2} = C.$



Pendulum: 势能: $-mgh = -mgR\cos\theta$



速度: $R\dot\theta$

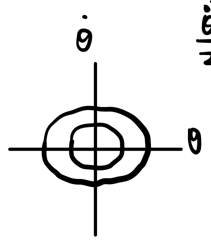
动能: $\frac{m(R\dot\theta)^2}{2} = \frac{m}{2}R^2\dot\theta^2$

能量守恒 $\frac{m}{2}R^2\dot\theta^2 - mgR\cos\theta = C$

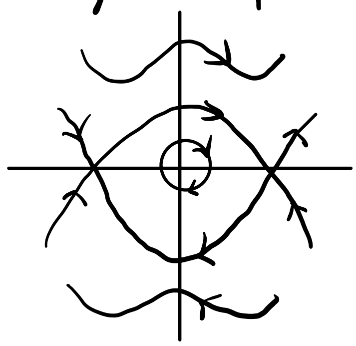
估算: $\cos\theta = 1 - \frac{\theta^2}{2}$ near $\theta = 0$.

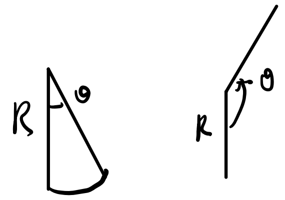
$$\frac{\dot\theta^2}{2} - \frac{g}{R}\cos\theta = \text{const}.$$

$$\frac{\dot\theta^2}{2} + \frac{g}{R}\theta^2 = \text{const}.$$



For physical pendulum:



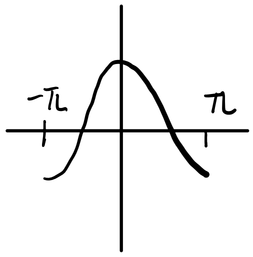


$$E(\theta, \dot\theta) = \frac{\dot\theta^2}{2} - \frac{g}{R}\cos\theta$$

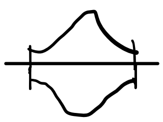
$E(\theta,\dot\theta) = C$ $\quad v=\dot\theta \quad$ $E(\theta, v) = C$ conservation of energy

plot level curves for E to obtain phase portrait.

$$\frac{v^2}{2} - \frac{g}{R}\cos\theta = C. \qquad v = \pm\sqrt{\underbrace{\frac{2g}{R}\cos\theta}_{f(\theta)} + C}.$$



I $\quad C > \frac{2g}{R}$



$\pm\sqrt{f(\theta)+C}$

II $\quad C = \frac{2g}{R}$

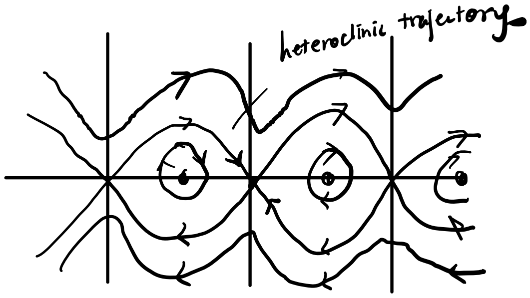
$f(\theta)+C$ $\qquad \pm\sqrt{f(\theta)+C}$

III. $C \in \left(-\frac{2g}{R}, \frac{2g}{R}\right)$

$f(\theta)+C$

IV $\quad C = -\frac{2g}{R}$

$\sqrt{\cdots}$



Combine all four.

Local theory of critical points.

Let $F: \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field

Def $p \in \mathbb{R}^n$ is called a critical stationary point if $F(p) = 0$.

$x(t) \equiv p$ is a solution of $\dot{x} = F(x)$

Taylor expansion near p.

$F(x) = F(p) + DF(p)(x-p) + o(|x-p|)$.

$DF(p) = $ Jacobian matrix at p. $= \begin{pmatrix} \partial_1 F_1(p) & \cdots & \partial_n F_1(p) \\ \partial_2 F_1(p) & \ddots & \partial_n F_2(p) \\ \vdots & & \\ \partial_n F_1(p) & \cdots & \partial_n F_n(p) \end{pmatrix}$

$\dot{y} = \dot{x} = F(x) = Ay + o(y)$.

$\dot{y} = Ax$.

It would be great if solutions of those two are well defined

$\dot{x} = x^2$. $\rightarrow \bullet \rightarrow$ $F'(x) = 2x$. $F'(0) = 0 = A$

linear $\dot{x} \equiv 0$.

Def. A critical point p for F is called hyperbolic if all eigenvalues of $DF(p)$ have non-zero real part.

Hartman-Grobman Thm

If A has no evalues with 0 real part.

$y = H(x)$. $\dot{x} = F(x) = A(x-p) + o(x-p)$

$\dot{y} = Ay$

One can always find "rectifying"

change of variables

<u>HG Thm:</u> Let F a C^1 vector field in $\mathbb{R}^n$, p is a hypobolic critical point. $A = DF(p)$

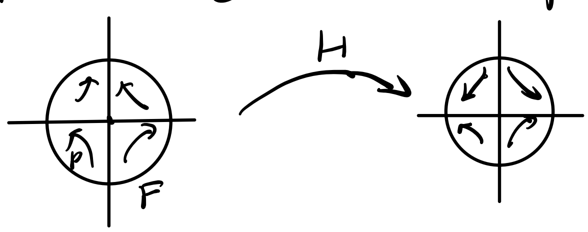
Then, there're 1) small neighb U of p.
2) small neighb V of 0.
3) homiomorphism $U \to V$. continuous from $U \to V$ also continuous from $V \to U$.

S.t. $H(p) = 0$ and the following holds:
Let $I \subset \mathbb{R}$ be a time interval ???

Stable / Unstable manifolds of a critical point.

Suppose p is a critical pt of a C^k vector field F in $\mathbb{R}^d$.

Hartman-Groben thm: if p is hypobolic



$A = DF(p)$.

$\dot{x} = Ax$.

$$A = DF(p) = \begin{pmatrix} \partial_1 F_1(p) & \cdots & \partial_n F_1(p) \\ \vdots & & \vdots \\ \partial_1 F_n(p) & \cdots & \partial_n F_n(p) \end{pmatrix} \qquad F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$$

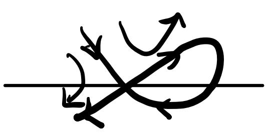
Φ: non-linear flow associated with F.

Φ_A: linear flow associated with matrix A, $\Phi_A = e^{tA}$.

$$\begin{tikzcd} U \arrow[r, "H"] \arrow[d, "\Phi^t"'] & V \arrow[d, "\Phi_A^t = e^{tA}"] \\ U \arrow[r, bend left, "H"] & V \arrow[l, bend left, "H^{-1}"] \end{tikzcd}$$

$$\Phi_A^t \circ H = H \circ \Phi^t$$

$$\Phi^t = H^{-1} \circ \Phi_A^t \circ H.$$



$$E^s = \left\{ \begin{array}{l} \text{space spanned by} \\ \text{Jordan basis vectors} \\ \text{with } Re < 0 \end{array} \right\} = \left\{ \begin{array}{l} x \in \mathbb{R}^n \\ \\ \lim_{t\to\infty} \Phi^t x = 0 \end{array} \right\}.$$

$$\underbrace{e^{\lambda t},\ e^{\lambda t}\cos}_{\to 0},\ \lambda < 0$$

$$E^u = \left\{ \begin{array}{l} \text{space spanned by} \\ \text{Jordan basis vectors} \\ \text{with } Re > 0 \end{array} \right\} = \left\{ \begin{array}{l} x \in \mathbb{R}^n \\ \lim_{t\to-\infty} \Phi_A^t x = 0 \end{array} \right\}.$$

E^s, E^u stable and unstable spaces for the linear flow

$W^s = \{x : \lim_{t\to\infty} \Phi^t x = p\}$.

$W^u = \{x : \lim_{t\to\infty} \Phi^t x = p\}$.

$H^{-1}(E^s \cap V) = W^s \cap V$

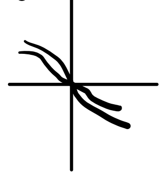
$H^{-1}(E^u \cap V) = W^u \cap V$

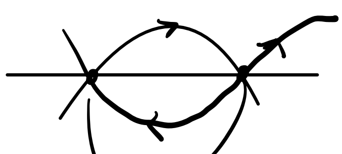
W^s and W^u can overlap.

$\mathbb{R}^n = E^s \oplus E^u \qquad x = \underset{\substack{\uparrow \\ E^s}}{x_s} + \underset{\substack{\uparrow \\ E^u}}{x_u}$

Stable / Unstable manifold theorem:

The sets W^s & W^u are actually as smooth as F is
($F \in C^k$, then W^s, W^u are obtained from E^s, E^u by C^k-smooth maps).





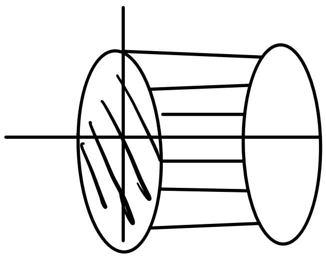
hypobolic critical points

the --- trajectories connect two saddles.

What about local behaviors near noncritical pts?

Def A point p is called regular for a vector field F if it is not critical, i.e. $F(p) \neq 0$

Archetypal example: $F(x) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for all x.

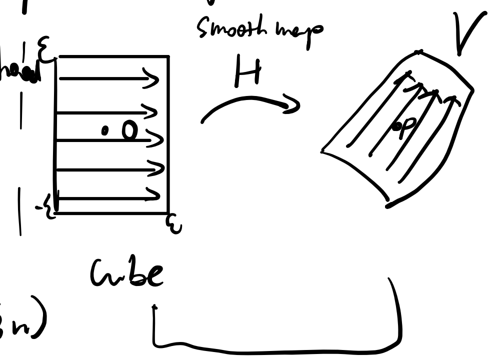


Tubular Flow.

$$\dot{x} = F(x) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$x(t) = x_0 + \begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = x_0 + t\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Tubular flow thm: Let p be a regular point of a vector field $F \in C^k$. Then there is $\varepsilon > 0$, a neighborhood V of p and a C^k-diffeomorphism ①
$H: (-\varepsilon, \varepsilon)^n \to V$, s.t. $H(0) = p$ and for all $\xi_1, \dots, \xi_n \in (-\varepsilon, \varepsilon)$, the function $H(t, \xi_2, \xi_3, \dots, \xi_n)$, $t \in (-\varepsilon, \varepsilon)$ is a solution of $\dot{x} = F(x)$.



Similar flows for non-critical pts, by change of coordinates

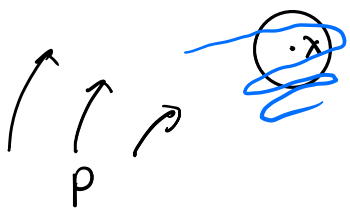
①. injective, partial derivative is in C^k.
diffeomorphism: differentiable bijection.

Limit Sets:

$\dot{x} = F(x)$ on $\mathbb{R}^2$. Flow Φ^t is complete. For $\forall\, p \in \mathbb{R}^2$, $\forall\, t \in \mathbb{R}$. $\Phi^t p$ is well defined.

eventually arrives when $t \to \infty$ / last thing dynamical system do.

$W(p) = \{x \in \mathbb{R}^2 : \exists\, (t_n)$ with $t_n \to \infty$ s.t. $\lim_{n\to\infty} \Phi^{t_n} p = x\}$



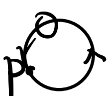
In other words, $W(p)$ consists of pts x s.t. every neighbor of x is visited infinitely many times.

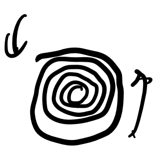
$\alpha(p) = \{x \in \mathbb{R}^2 : \exists\, (t_n)$ with $t_n \to -\infty$ s.t. $\lim_{n\to\infty} \Phi^{t_n} p = x\}$.

① If p is critical point, $F(p) = 0$. then $W(p) = \{p\}$.
$\alpha(p) = \{p\}$

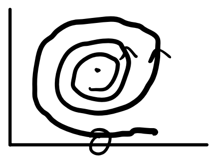
②. If $\lim_{t\to\infty} \Phi^t p = x$, then $W(p) = \{x\}$

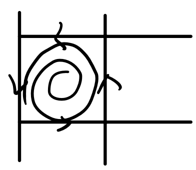
∴ attractive cycle ∴

③ limit cycle: 

 limit cycle $W(p) = \{$ periodic trajectory $\}$.

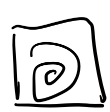
④ close to x axis



⑤. 

Thm Assume that p satisfies: $\exists$ compact set K s.t. $\Phi^t p \in K$ all $t \geq 0$, then $W(p)$ is

positive orbit

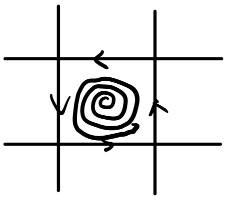


1) invariant under (Φ^t)

2) compact

3) connected

Def: A set B is forward (backward) invariant under (Φ^t) if for all $x \in B$, all $t \geq 0$ ($t \leq 0$) $\Phi^t x \in B$



never exceed

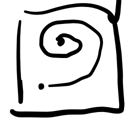
Formal: foward + backward = invariant

Proof. in pdf lecture notes

Poincaré - Bendixson Thm:

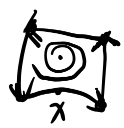
<u>Thm</u> Let F be C^1 vector field on $\mathbb{R}^2$ with finitely many critical points. Suppose p is such that $\{\varphi^t p, t \geq 0\}$ is contained in a component set K. Then,

1). If $\omega(p)$ contains <u>only</u> critical points, then $\omega(p) = \{$one critical point$\}$.



2). If $\omega(p)$ contains only regular points, then $\omega(p) = \{$one periodic orbit$\}$.

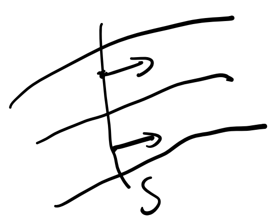
3). If $\omega(p)$ contains reguler and critical pts. then for each regular $x \in \omega(p)$



$\omega(x) = \{$one critical point$\}$.

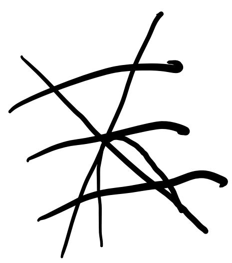
$\alpha(x) = \{$one critical pt$\}$.

Let x be regular for F



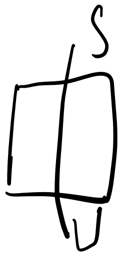
Can find a transverse section S near x

$S =$ curve: $F(y)$ not tangent to S at all pts y.



Fact 1: If x is regular, $x \in \omega(p)$, then we can find $t_n \to \infty$ s.t. $\varphi^{t_n} p \in S$.

<u>proof</u>. Since $x \in \omega(p)$, come back to small neighborhood of x infinitely many times $s_1, s_2, s_3, \dots$



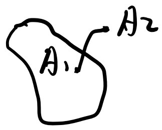
Use tubular flow thm to adjust their $s_1 \mapsto t_1$, $s_2 \mapsto t_2$ ...

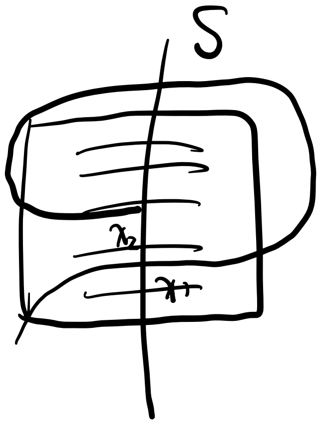
Addition to Fact 1): $\exists\, \delta > 0$, $t_{n+1} - t_n > \delta$. Let $x_n = \Phi^{t_n} p$

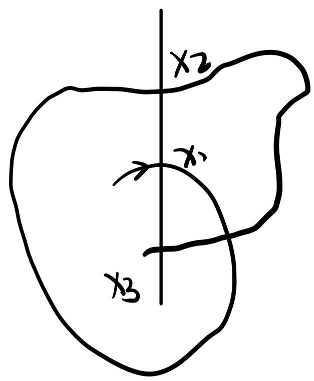
Fact 2): x_n is a monotone sequence in S.

Jordan's planar curve thm:

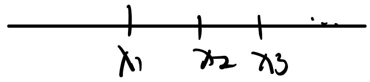
A closed simple curve divides the plane into two connected (no pts of) components A_1, A_2.







Fact 3). We already know $x \in \omega(p) \cap S$. There're no other points in $\omega(p) \cap S$ (A monotone sequence can have only one limit point)



Fact 4. Let $x \in \omega(p)$ If $\omega(x)$ or $\alpha(x)$ contains a regular pt, then x is a periodic pt and $\omega(p) = \alpha(x) =$ {trajectory of x}.

Proof. $\omega(x) \cup \alpha(x) \subset \omega(p)$

$x \in \omega(p) \Rightarrow$ (because $x \in \omega(p)$, $\omega(p)$ is invariant and compact).

Let $y \in \omega(x) \cup \alpha(x) \subset \omega(p)$. regular.

Then $y \in \omega(p)$, S the transverse section.

Then (A). $\omega(p) \cap S = \{y\}$.

(B). $\Phi^t x$ comes back to neighborhood of y infinitely many times

(A), (B) $\Rightarrow$ must come back exactly to y.

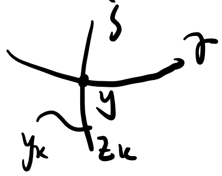
This means that $\Phi^t x$ is a periodic trajectory passing through y

So x is periodic, $\gamma = \omega(x) = \alpha(x) = \{\text{traj of } x\}$.

Suppose $\omega(p) \setminus \gamma \neq \phi$

$\omega(p)$ is connected

$\exists\ y_k \to y$
$\ \ \omega(p)\setminus\gamma \quad \gamma$



Adjusting y_k we obtain $\exists\ z_k \to y$ $(\omega(p)\setminus\gamma) \cap S \subset \omega(p) \cap S$

But $\omega(p) \cap S$ contains only one point y. (Fact 3).

So, $z_k = y$ $(\exists k)$ $\quad \omega(p)\setminus\gamma \neq \phi$ is wrong.

So $\omega(p) = \gamma$.

Def. A critical point x_0 is (Lyaponov) stable if for every $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |\Phi^t x - x_0| < \varepsilon$ for all t.

Def. x_0 is called asymbolically stable if

1). it's stable [trap]

2). $\exists\, \delta$. $|x - x_0| < \delta \Rightarrow \Phi^t x \to x_0$,

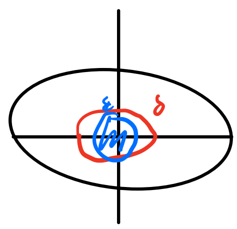
$|\Phi^t x - x_0| \to 0$. $\leq Ce^{-ut}$, $u > 0$

Def. x_0 is exponentially stable if convergence is exponetial.

Ex. $\dot{x} = -\lambda x$.

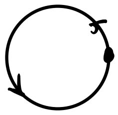
$x(t) = x(0) e^{-\lambda t}$ $\quad |x(t) - 0| \leq e^{-\lambda t} |x(0)|$

Lyoponov stability not asympototic

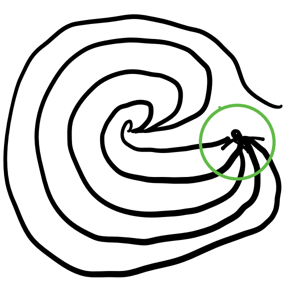


Def. marginally stable.

ODE on the circle $= S^1$



$\longrightarrow$ critical point is attractive. (eventually goes there).



attractive but not Lyponov stable.

Thm: Let $F \in C_1$ near a hyperbolic critical pt x_0. Then

1) all the real parts of eigenvalues are negative
$\Rightarrow$ asymptotically stable

2). if not 1).
$\Rightarrow$ not Lyponov stable.

Proof. H-G: change of veriebles $x = H(y)$
$\dot{y} = Ay$. $A = DF(x_0)$

The notion of lyoponov and asympot are preserved by this transformation,

$y(t)$ = linear combination of $t^k e^{\lambda t}$, $t^k e^{at} \cos(bt)$, $t^k e^{at} \sin(bt)$

λ = e.v. $a+ib$ = e.v. has to be negative!

Lyapunov Functions:

EX $\dot{x}_1 = -\frac{1}{2}x_1 - \frac{1}{2}x_2$ $\quad \dot{x} = Ax$. 0 is c.p.t.

$\dot{x}_2 = \frac{1}{2}x_1 - \frac{1}{2}x_2$

instead of solving

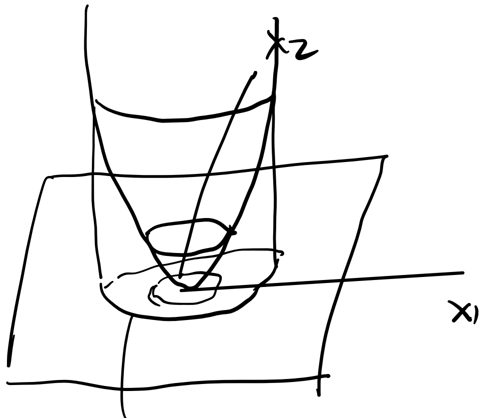
$V(x) = x_1^2 + x_2^2$ $\quad \frac{d}{dt}V(x(t)) = \frac{d}{dt}(x_1(t)^2 + x_2(t)^2)$

$= 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$= 2x_1(-\frac{1}{2}x_1 - \frac{1}{2}x_2) +$

$= -x_1^2 - x_2^2$

< 0. if $x(t) \neq 0$.



shrinking $\frac{dV}{dt} < 0$.

$\dot{x} = F(x)$

Let $V: \mathbb{R}^d \to \mathbb{R}$ is C^1

$$\frac{d}{dt} V(x(t)) = \frac{d}{dt} V(x_1(t), x_2(t), \dots, x_d(t)).$$

chain rule: $\partial_1 V(x(t)) \underbrace{\dot{x}_1(t)}_{=F_1(t)} + \dots + \partial_d V(x(t)) \underbrace{\dot{x}_d(t)}_{=F_d(t)}$

$$= \sum_{i=1}^{d} \partial_i V(x(t)) F_i(x(t)).$$

$$= \langle \nabla V(x(t)), F(x(t)) \rangle$$

It would be nice if $\langle \nabla V(x), F(x) \rangle < 0$ for all $x \neq x_0$

Def. $V: \mathbb{R}^d \to \mathbb{R}$ is a lyapunov function for $\dot{x} = F(x)$ and a critical point $x_0 = 0$ if

1). $V \in C^1$

2). $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$.

3). $\langle \nabla V(x), F(x) \rangle \leq 0$ for all $x \neq 0$

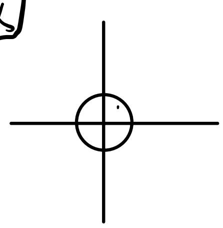
Def. If addition $\langle \nabla V(x), F(x) \rangle < 0$, then strict Lyapunov function.

Lypunov Stability Theorem:

1). If V is a L.f. Then 0 is Lypunov stable.

2). If V is a strict L.f. then 0 is asymptotically stable.

$\boxed{\text{Ex}}$



$\frac{d}{dt} V(x) = 0$.

Satisfies 1).

Check Lyapunov !!