HW1-FS

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1 Hedging in a 1-Period Trinomial Model

1.1 Basic Model Considerations

1.1.1 (a) Is the Model arbitrage free?

In order to check if the model is arbitrage-free, the no-arbitrage condition should be satisfied. This condition stipulates that the expected return of the underlying under the real-world probability measure should be equal to the risk-free rate. Mathematically, the no-arbitrage condition can be expressed as:

$$E_P[S_T] = Se^{rT}$$

where $E_P[S_T]$ is the expected value of S_T under the real-world probability measure P. We compute this by taking the weighted sum of possible future values:

$$E_P[S_T] = p_1 S_1 + p_2 S_2 + p_3 S_3$$

Thus, We have to check if:

$$p_1 S_1 + p_2 S_2 + p_3 S_3 = Se^{rT}$$

If this equation holds, then the model is arbitrage-free. Otherwise, it allows for arbitrage opportunities.

1.1.2 (b) How many risk neutral measure are there?

A martingale measure (or risk-neutral measure) Q is a probability measure under which the price of an asset is expected to be equal to its current price discounted at the risk-free rate.

$$E_Q[S_T] = S_0$$

where $E_Q[S_T]$ is the expected value of S_T under measure Q.
In a discrete model like this one, we have a finite set of possible outcomes for S_T. We can find a risk-neutral measure by finding probabilities p_1, p_2, p_3 under Q such that

$$p_1 S_1 + p_2 S_2 + p_3 S_3 = S$$

and

$$p_1 + p_2 + p_3 = 1$$

If there is a unique set of probabilities (q_1, q_2, q_3) that satisfies the above conditions, then there is a unique martingale measure. If there are multiple sets of such probabilities, then there are multiple martingale measures

1.1.3 (c) Is the market complete?

A market is said to be complete if every contingent claim (i.e., a financial derivative) can be perfectly replicated by a portfolio consisting of the underlying asset and the risk-free asset. In other words, if there exists a unique martingale measure, the market is complete.

In this 1-period model with three possible future values of the underlying, since the system has three outcomes and we can construct a portfolio with at most two independent assets (the risk-free asset and the underlying), there is a unique way to hedge any derivative if and only if there is a unique martingale measure. So, if there is a unique martingale measure, then the market is complete. Otherwise, it is incomplete.

2 Mean-Variance Hedging

2.1

Let's imagine you sold a derivatives with (generic) payoff $g(S_T)$ that you will hedge with a self-financing portfolio with initial endowment p and quantity of underlying Δ. We note $P_T = P_T(p, \Delta)$ the value of the hegding portfolio at time T. To hedge our option, we want to minimize

$$\varepsilon(p, \Delta) = E\left[(P_T - g(S_T))^2\right]$$

that is, we want to find the initial portfolio value and delta-hedging strategy that minimizes our quadratic risk.

2.1.1 What is the expression P_T of the self-financing portfolio at time T ?

In a self-financing portfolio under a risk-neutral measure, the portfolio value at time T, denoted as P_T, evolves according to:

$$P_T = \Delta S_T + (p - \Delta S_0)e^{rT}$$

2.1.2 What are the optimal (p, Δ) that minimize $\varepsilon(p, \Delta)$?

The objective is to find (p, Δ) that minimizes the quadratic risk:

$$\varepsilon(p, \Delta) = E\left[(P_T - g(S_T))^2\right]$$

To find the optimal (p, Δ), we can solve the following optimization problem:

$$\min_{p,\Delta} \varepsilon(p, \Delta)$$

We take the partial derivatives of $\varepsilon(p, \Delta)$ with respect to p and Δ, and set them equal to zero, forming a system of equations to solve for p and Δ. The partial derivatives are given by:

$$\frac{\partial \varepsilon(p, \Delta)}{\partial p} = E\left[2\left(\Delta S_T + (p - \Delta S_0)e^{rT} - g(S_T)\right) \cdot (-e^{rT})\right] = 0$$

$$\frac{\partial \varepsilon(p, \Delta)}{\partial \Delta} = E\left[2\left(\Delta S_T + (p - \Delta S_0)e^{rT} - g(S_T)\right) \cdot (S_T - S_0 e^{rT})\right] = 0$$

Given two equation with two variables, we can solve for the optimal p, Δ as

$$\Delta = \frac{Cov[S_T - S_0 e^{rT}, g(S_T)]}{Var[S_T - S_0 e^{rT}]}$$

$$\begin{aligned} p &= e^{-rT}(E[g(S_T)] - \frac{E[S_T - S_0 e^{rT}] \cdot Cov[S_T - S_0 e^{rT}, g(S_T)]}{Var[S_T - S_0 e^{rT}]}) \\ &= e^{-rT}(E[g(S_T)] - \Delta E[S_T - S_0 e^{rT}]) \end{aligned}$$

2.1.3 What is the expression of residual error, namely $\varepsilon(p, \Delta)$ for the optimal solution?

Subsitute the previous result into the expression

$$\begin{aligned} \varepsilon(p, \Delta) &= E\left[(P_T - g(S_T))^2\right] \\ &= E\left[\left(\Delta S_T + (p - \Delta S_0)e^{rT} - g(S_T)\right)^2\right] \\ &= \Delta^2 Var[S_T - S_0 e^{rT}] + Var[g(S_T)] \\ &= \frac{Cov[S_T - S_0 e^{rT}, g(S_T)]}{Var[S_T - S_0 e^{rT}]} + Var[g(S_T)] \end{aligned}$$

2.2 (b)

Find a martingale measure $Q = (q_1, q_2, q_3)$ which does not depend on g (at all) such that

$$p = E^Q \left(g\left(S_T\right) e^{-rT} \right) = \sum_i g\left(S_i\right) e^{-rT} q_i$$

To find a martingale measure $Q = (q_1, q_2, q_3)$ that does not depend on the payoff function $g(S_T)$, we should consider the two conditions:

1. The discounted future asset price's expected value must be equal to its current price under measure Q:

$$E_Q[S_T e^{-rT}] = S_0$$

$$q_1 S_1 e^{-rT} + q_2 S_2 e^{-rT} + q_3 S_3 e^{-rT} = S_0$$

Thus, a martingale measure $Q = (q_1, q_2, q_3)$ satisfies that

$$(q_1, q_2, q_3) \cdot (S_1, S_2, S_3) = S_0 e^{rT}$$

2. The probabilities must sum to one and be non-negative:

$$q_1 + q_2 + q_3 = 1$$

$$q_1, q_2, q_3 \geq 0$$

2.3 (c) Show that $\sum_i q_i = 1$

The total probability assigned to all possible outcomes should equal 1. This comes directly from the fundamental axiom of probability that the sum of the probabilities of all possible outcomes in the sample space must equal 1. Symbolically,

$$\sum_i q_i = q_1 + q_2 + q_3 = 1$$

In pricing derivatives, ensuring $\sum q_i = 1$ is also critical for ensuring no arbitrage in the model and for the model to be economically reasonable. For example, if $\sum q_i > 1$, then we could make an arbitrage profit by taking a short position in the derivative, and if $\sum q_i < 1$, we could make an arbitrage profit by taking a long position in the derivative.

2.4 (d) Compute $E^Q\left(S_T\right)$

The expectation $E^Q(S_T)$ under the risk-neutral measure Q can be computed as follows, given that S_T can take values S_1, S_2, and S_3 with respective probabilities q_1, q_2, and q_3 under Q:

$$E^Q(S_T) = q_1 S_1 + q_2 S_2 + q_3 S_3$$

$E^Q(S_T)$ represents the expected value of the asset price at time T under the risk-neutral measure Q. It is a theoretical expectation used for pricing assets and derivatives in a consistent, arbitrage-free manner.

2.5 (e) Show that $0 < q_i < 1$ for all i

Probabilities are always bounded by 0 and 1, according to the axioms of probability. A probability of 0 indicates impossibility, while a probability of 1 indicates certainty. Any value outside the $[0,1]$ interval would violate the definition of probability. In this case, since all three probability represents a future outcome, they cannot be 0
Next, we prove that q_i cannot take 1. Suppose, for the sake of contradiction, some $q_i = 1$ and the rest $q_j = 0$. Then, by the result from part b, we will have $S_i = Se^{eT}$, which contradicts the our assumption $S_1 < S_2 < Se^{rT} < S_3$. Therefore, $0 < q_i < 1$ for all i.

3 Comparison with Binomial Tree

We set $S = 100, S_1 = 92, S_2 = 98, S_3 = 105, r = 0$ and $P(S_T = S_i) = p_i$ with $(p_1, p_2, p_3) = (1/3, 1/4, 5/12)$ and assume that we sold a put option with strike $K = 100$. (a) What are the values of p and Δ in this specific trinomial model?
(b) Because $S_2 < Se^{rT} < S_3$, we think of keeping $S_3 = 105$ the same and look for a value $\tilde{S}_1$ such that $\Delta_{\text{bin}} = \Delta$ and the binomial model would have two possible outcomes, namely $\tilde{S}_1$ that needs to be computed and $S_3 = 105$. i. Explain why there is a solution $\tilde{S}_1$ such that $\Delta_{bin} = \Delta$ and how to compute it.
ii. What is the value of p_{bin} ?
(c) We assume that in reality, the stock realized at 92 , that is $S_T = 92$, which was a possible value in the trinomial model. The P& L of the hedge position is the difference between the value of the hedging portfolio and the derivatives payoff (we are supposed to pay to our client), that is $P\&L = P_T - (K - S_T)^+$.
i. What if the P&L of the hedged position if we used the binomial model?
ii. What if the P&L of the hedged position if we used the trinomial model?
iii. What do you conclude?
(d) We now consider "out-of-sample" scenarii for the first model:
i. What if the underlying drops even more, namely $S_T = 90$: what is the P&L from using each model?
ii. What if the underlying goes up to $S_T = 105$: what is the P&L from using each model?
(e) What if instead we choose $\tilde{S}_1$ so that $p_{\text{bin}} = p$.
i. How to compute the solution for such $\tilde{S}_1$ in general and what is its numerical value in this specific example?
ii. What is the corresponding Δ_{bin} ?
iii. Compare the P&L (in a table) of this model for each previous senarii against the trinomial model.
(f) Summarize in a table the P&L of each model for all the previous scenarii (and feel free to add some, best is to code it and show a graph).

3.1 (a)

The payoff of the put option at expiration is $max(K - S_T, 0)$.
1. For $S_1 = 92$: $g(S_1) = max(100 - 92, 0) = 8$
2. For $S_2 = 98$: $g(S_2) = max(100 - 98, 0) = 2$
3. For $S_3 = 105$: $g(S_3) = max(100 - 105, 0) = 0$
And we have

$$E[g(S_T)] = \frac{1}{3} \times 8 + \frac{1}{4} \times 2 + \frac{5}{12} \times 0 = \frac{8}{3} + \frac{1}{2} = \frac{19}{6}$$

$$E[S_T - S] = \frac{1}{3} \times (-8) + \frac{1}{4} \times (-2) + \frac{5}{12} \times 5 = \frac{-8}{3} - \frac{1}{2} + \frac{25}{12} = \frac{-13}{12}$$

$$Var[S_T - S] = \frac{1}{3} \times 64 + \frac{1}{4} \times 4 + \frac{5}{12} \times 25 - (\frac{13}{12})^2 = \frac{4547}{144}$$

$$Cov(S_T - S, g(S_T)) = \frac{1}{3} \times (-64) - \frac{1}{4} \times 4 + \frac{5}{12} \times 0 + \frac{19}{6} \times \frac{13}{12} = -\frac{1361}{72}$$

Using the formula computed above, we have

$$\Delta = -\frac{1361}{72} \times \frac{144}{4547} \approx -0.599$$

$$p = \frac{19}{6} - \Delta\frac{-13}{12} \approx 2.52$$

3.2 (b)

3.2.1 (i)

If we set $\Delta_{bin} = \Delta$, it means that the hedging strategy in both the trinomial and binomial models will be the same. This is because Δ provides the amount of the stock we need to hold in our hedging portfolio to be indifferent to small changes in the stock price. In the binomial model, by varying $\tilde{S_1}$, we can adjust the hedging strategy. Since Δ gives the sensitivity of the option's price to changes in the underlying stock's price, there exists a value of $\tilde{S_1}$ that will make the option's sensitivity in the binomial model match that in the trinomial model.

To find $\tilde{S_1}$, given that the payoff of the put option is $g(S) = \max(K - S, 0)$,

$$g(\tilde{S_1}) = \max(100 - \tilde{S_1}, 0)$$
$$g(S_3) = \max(100 - 105, 0) = 0$$

The binomial model will be:

$$p_{bin} = g(\tilde{S_1})q + g(S_3)(1 - q)$$

where q is the probability of reaching $g(S_3)$. Set Δ_{bin} to Δ:

$$\Delta = \frac{g(S_3) - g(\tilde{S_1})}{S_3 - \tilde{S_1}}$$

Plug in the values:

$$\Delta = \frac{0 - \max(100 - \tilde{S_1}, 0)}{105 - \tilde{S_1}} \approx -0.599$$

From the trinomial model, we can find the value of Δ and then solve for $\tilde{S_1}$.

$$\tilde{S_1} \approx 92.5$$

3.2.2 (ii)

From the above formula, we can calculate p_{bin}:

$$p_{bin} = g(\tilde{S_1})q + g(S_3)(1 - q)$$

with

$$q = \frac{1 - d}{u - d} = \frac{1 - 1.05}{\tilde{S_1}/100 - 1.05} = 0.4$$

Therefore, we have

$$p_{bin} \approx 7.5 \times 0.4 + 0 = 3$$

3.3 (c)

3.3.1 (i)

Binomial Model:

$$P_T = \Delta S_T + (P_0 - \Delta S_0)e^{rT} = 7.792$$

$$P\&L_{bin} = -8 + 7.792 = -0.208$$

3.3.2 (ii)

Trinomial Model:

$$P_T = \Delta S_T + (P_0 - \Delta S_0)e^{rT} = 7.312$$

$$P\&L_{tri} = -8 + 7.312 = -0.688$$

3.3.3 (iii)

Conclusion: The P&L is smaller for the trinomial model (one loses more), which means under this scenario the binomial model is a better choice.

3.4 (d)

3.4.1 (i)

Case when $S_T = 90$
Binomial Model:

$$P_T = \Delta S_T + (P_0 - \Delta S_0)e^{rT} = 8.99$$
$$P\&L_{bin} = -10 + 8.99 = -1.01$$

Trinomial Model:

$$P_T = \Delta S_T + (P_0 - \Delta S_0)e^{rT} = 8.51$$
$$P\&L_{tri} = -10 + 8.51 = -1.49$$

3.4.2 (ii)

Case when $S_T = 105$
Binomial Model:

$$P_T = \Delta S_T + (P_0 - \Delta S_0)e^{rT} = 0.005$$
$$P\&L_{bin} = 0.005 = 0.005$$

Trinomial Model:

$$P_T = \Delta S_T + (P_0 - \Delta S_0)e^{rT} = -0.475$$
$$P\&L_{tri} = 0 - 0.475 = -0.475$$

3.5 (e)

3.5.1 (i)

The binomial model will be:

$$p_{bin} = g(\tilde{S}_1)q + g(S_3)(1-q) = 2.52$$

where q is the probability of reaching $g(S_3)$.

$$q = \frac{1-d}{u-d} = \frac{1-1.05}{\tilde{S}_1/100 - 1.05}$$

With the above 2 equations we can solve for

$$\tilde{S}_1 \approx 94.92$$

3.5.2 (ii)

Further, we have

$$\Delta = \frac{g(S_3) - g(\tilde{S}_1)}{S_3 - \tilde{S}_1} = -0.504$$

3.5.3 (iii)

	S_T	$P\&L_{bin}$	$P\&L_{tri}$
1	92	-1.448	-0.688
2	90	-2.44	-1.49
3	105	0	-0.475

3.6 (f)

Graphical Representation

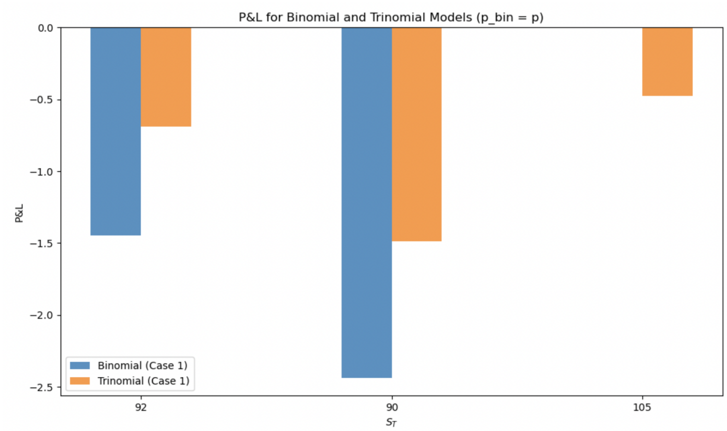

Figure 1: Graph representation for Case 1

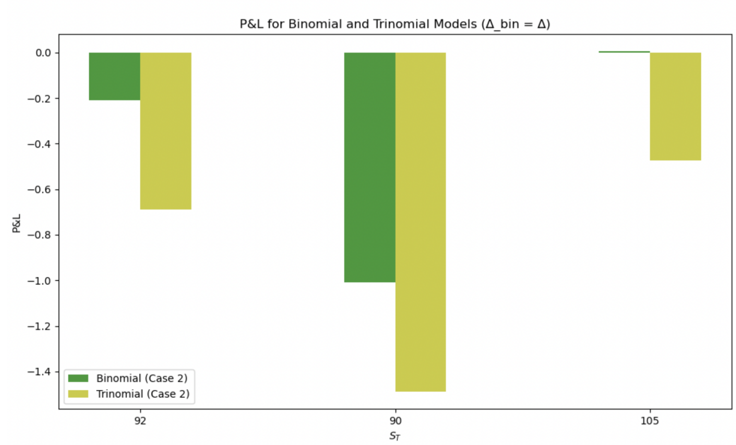

Figure 2: Graph representation for Case 2