$z = x + iy$, $z \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ $\boxed{\begin{array}{l} z \cdot 1 = x + iy \\ z \cdot i = -y + ix \end{array}}$

2D vector space

$$z_1 \cdot z_2 = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 & -(x_1y_2 - y_1x_2) \\ y_1x_2 + x_1y_2 & x_1x_2 - y_1y_2 \end{pmatrix}$$

$$\underbrace{z = (x+iy)}_{\text{Complex num}} \longleftrightarrow \underbrace{(x, y)}_{\mathbb{R}^2} \longleftrightarrow \underbrace{\begin{pmatrix} x & -y \\ y & x \end{pmatrix}}_{\text{Matrix.}}$$

$\det\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x^2 + y^2$

Any Complex number is invertible as long as it's not 0.

x called <u>real part</u>

y called <u>imaginery part.</u>

$|z| = \sqrt{x^2+y^2}$, $\bar{z} = x - iy. \Rightarrow x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$, $|z| = \sqrt{z \cdot \bar{z}}$

$\dfrac{z_1}{z_2} = z_2^{-1} \cdot z_1$, $\dfrac{1}{z} = \dfrac{1}{z} \cdot \dfrac{1}{\bar{z}} \cdot \bar{z} = \dfrac{1}{x+iy} \cdot \dfrac{1}{x-iy} \cdot (x - iy) = \dfrac{x-iy}{x^2+y^2}$

$a = \dfrac{x}{x^2+y^2}$, $b = \dfrac{-y}{x^2+y^2}$, $\dfrac{1}{z} = a + bi$

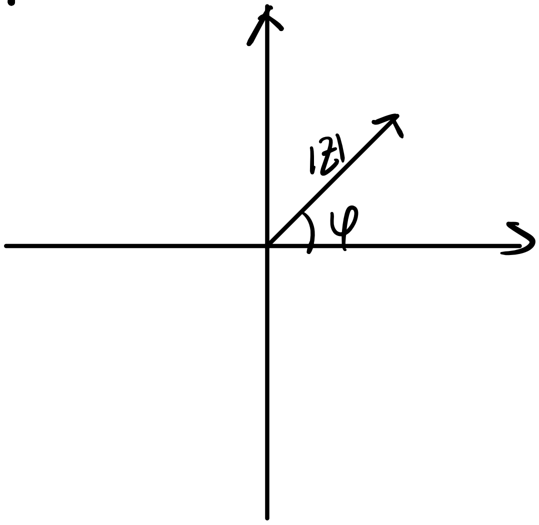
Power

$e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$ (exponetial function)

$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n+1}}{(2n+1)!}$

$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n}}{(2n)!}$

Polar Coordinate



$z = |z|(\cos\varphi + i\sin\varphi)$

General: $z = |z| e^{i\varphi}$

$z - \frac{z^2}{2!} - \frac{iz^3}{3!}$

Euler's formula: $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$

proof. $e^{i\varphi} = \sum_{n=0}^{\infty} \frac{(i\varphi)^n}{n!} = \frac{1}{1} + \frac{i\varphi}{1} - \frac{\varphi^2}{2!} - \frac{i^3\varphi^3}{3!} + \cdots$ (Series Addition)

$= \cos(\varphi) + i\sin\varphi$ ▣

Propeties:

① $e^z \cdot e^w = e^{z+w} \Rightarrow e^{it} e^{-it} = 1$, $t \in \mathbb{R}$.

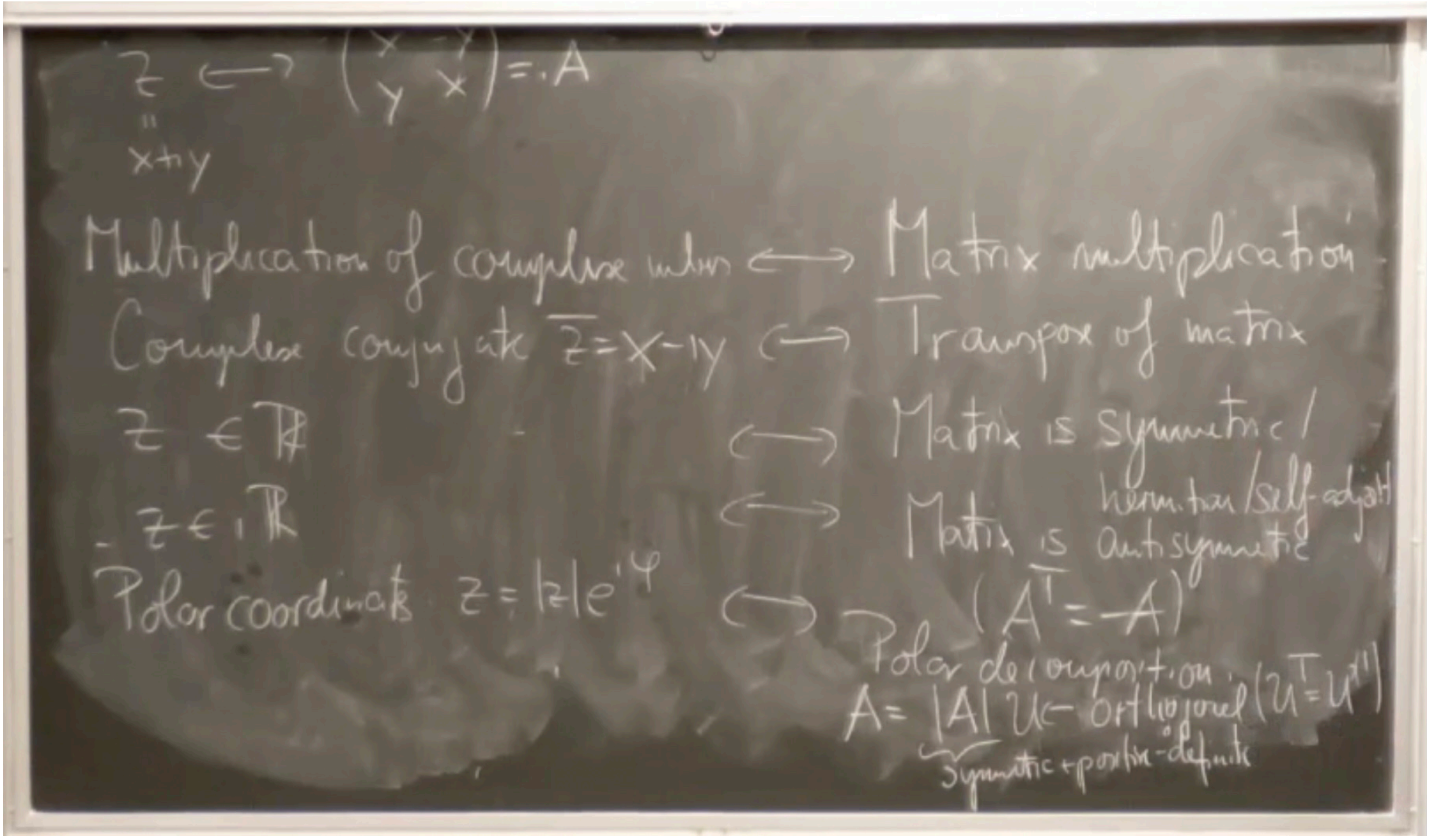
$\overline{e^{it}} = \sum_{n=0}^{\infty} \frac{\overline{(it)}^n}{n!} = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} = e^{-it}$

② $\overline{z+w} = \bar{z} + \bar{w}$

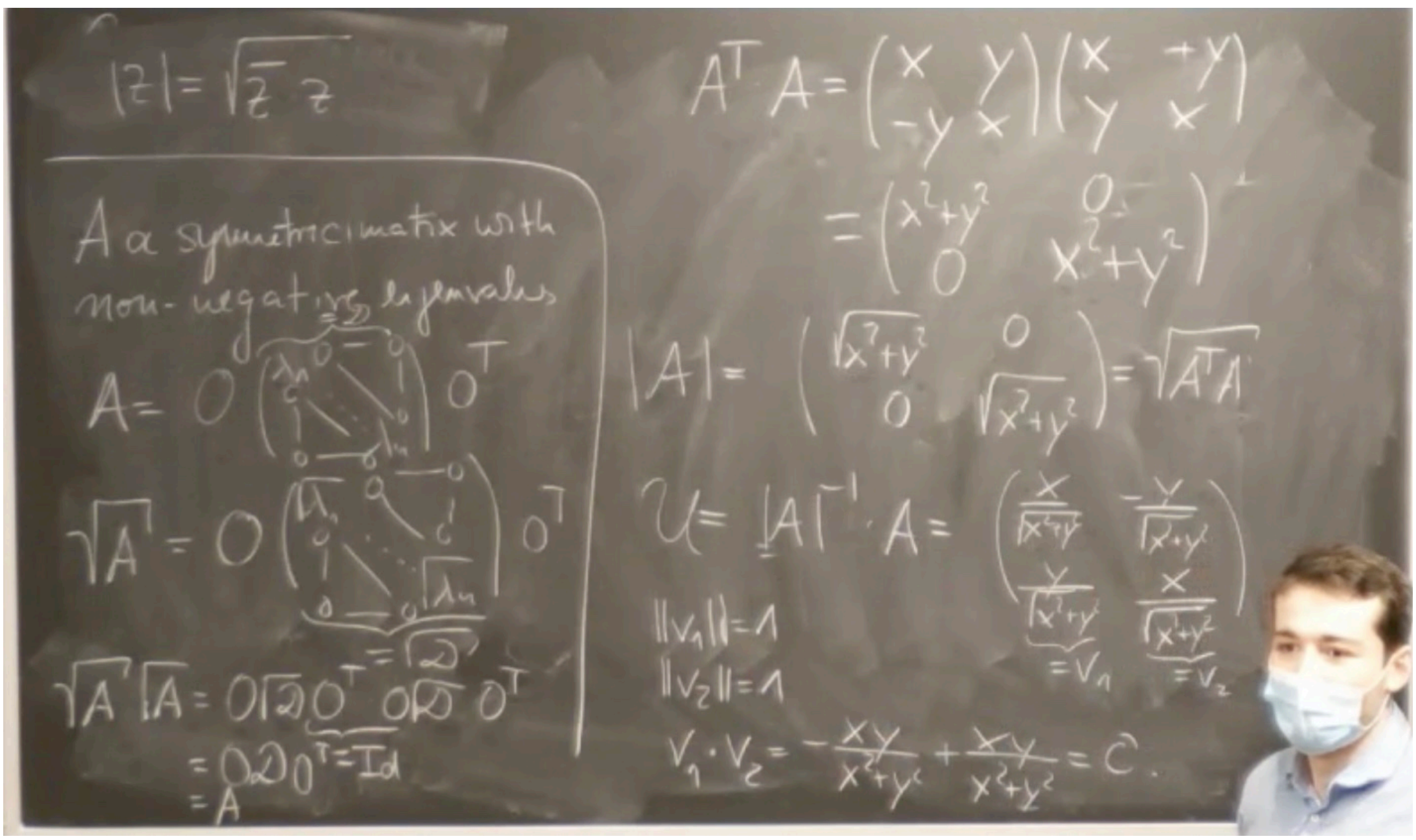
$\overline{zw} = \bar{z}\,\bar{w}$

proof. easy with matrices.

Concepts Relationships.



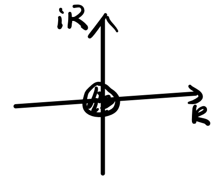
Explanations of last one:



Complex Differentiation

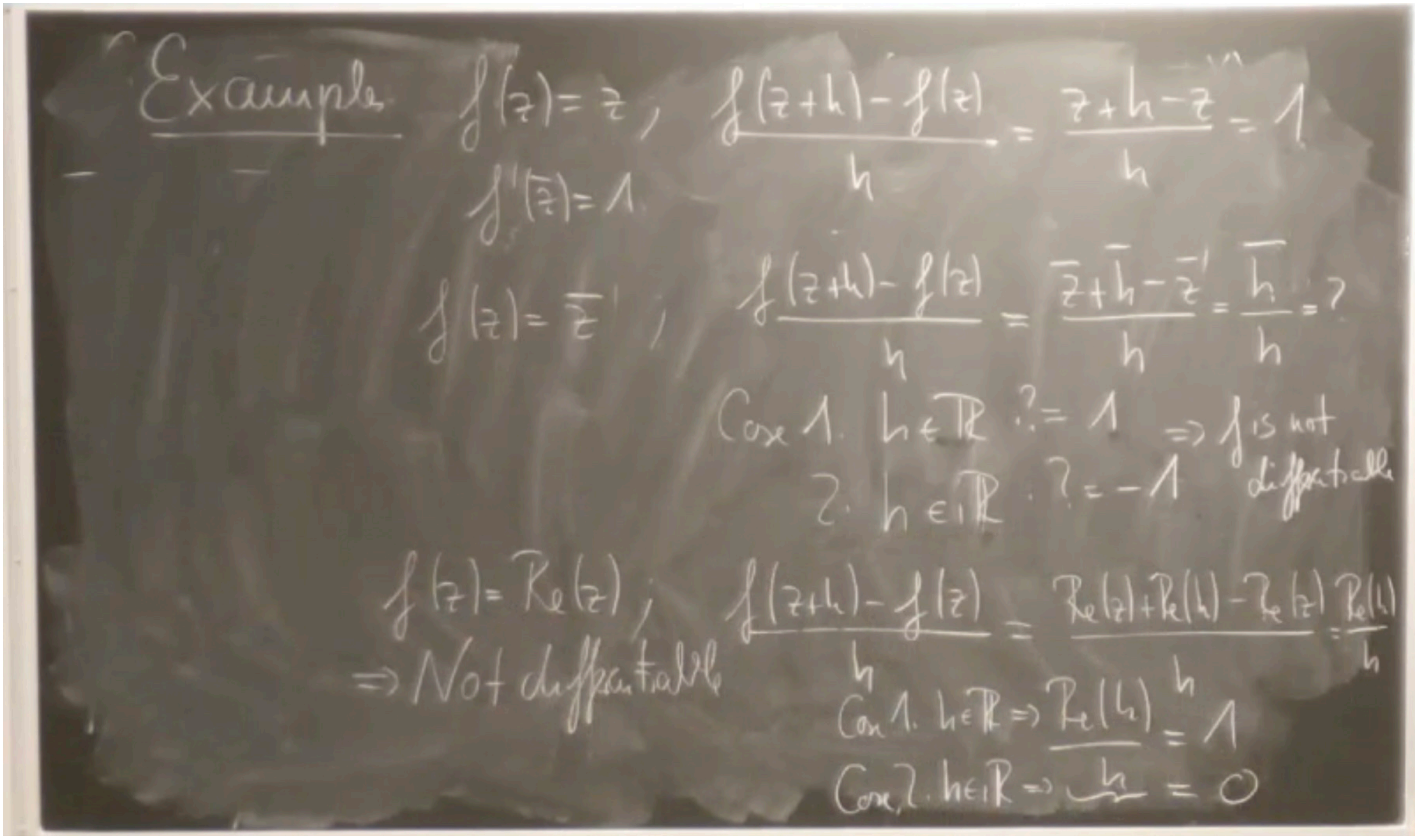
I.

Def. Let $f\ \mathbb{C}\to\mathbb{C}$, then we say f is differentiable at a point $z\in\mathbb{C}$ if $f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ exists.



All Directions

E.X.



II Def.

f is differentiable at $z \in \mathbb{C}$ iff exits γ, φ

$f(z+h)=f(z)+\gamma h+\varphi(z+h)$ where $\lim_{h\to 0}\frac{\varphi(z+h)}{h}=0$

$\varphi(z+h) \Leftrightarrow B_{\varepsilon}(z)=\{y\in\mathbb{C}: |z-y|<\varepsilon\}$, then $\gamma=f'(z_0)$

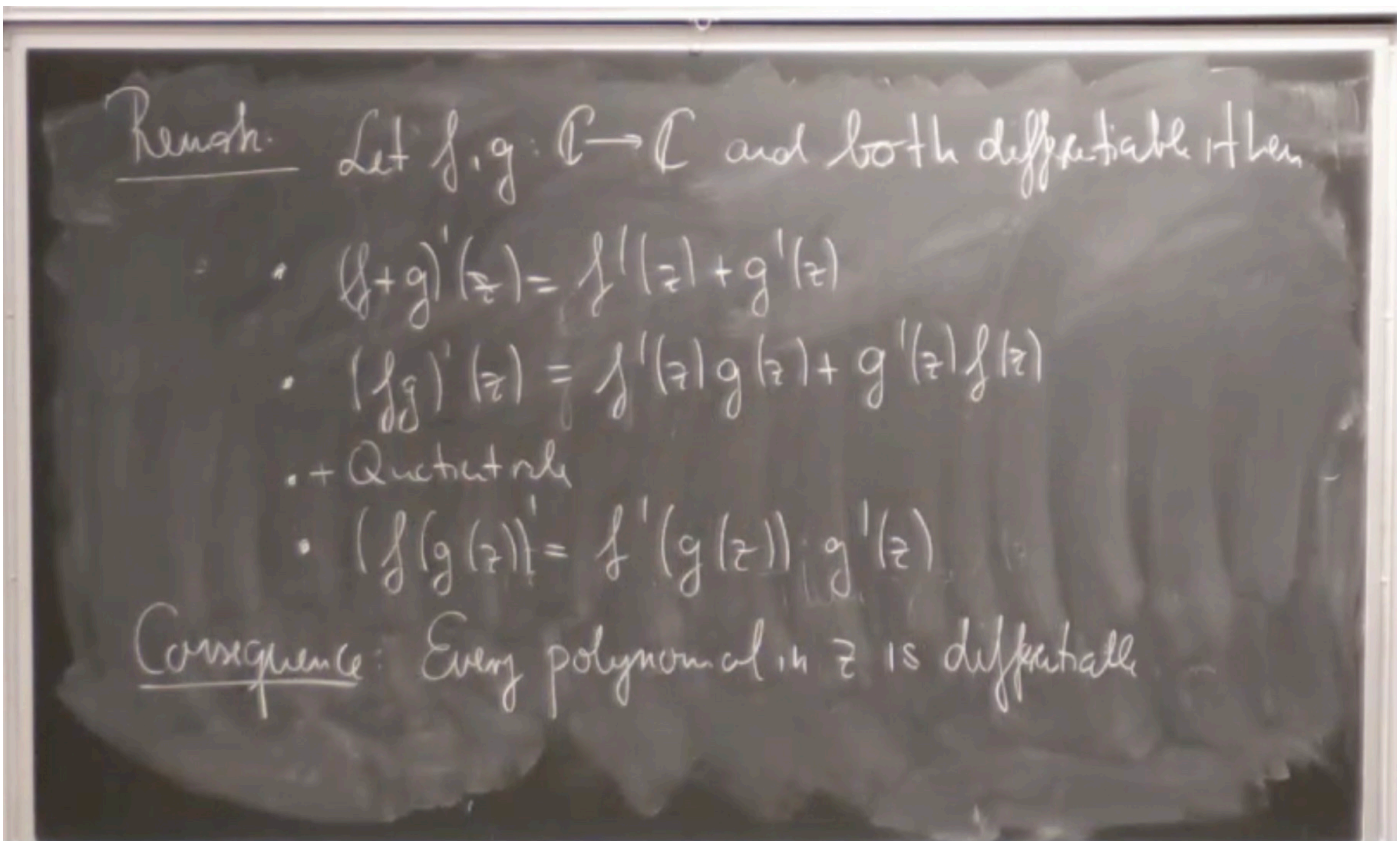
Proof.

$$\Rightarrow \lim_{h\to 0} \frac{\varphi(z+h)}{h} = \underbrace{\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}}_{f'(z)} - f'(z) = 0$$

$$\Leftarrow \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{\gamma h + \varphi(z+h)}{h} = \gamma + \lim_{h\to 0} \frac{\varphi(z+h)}{h} = \gamma$$

$$= f'(z)$$

Remarks:



Ways to check differentiability.

$f(z) = \bar{z}$, not differentiable.

Equivalent: $F: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad F(x,y) = (\underset{\text{Real}}{F_1(x,y)}, \underset{\text{Imaginary}}{F_2(x,y)})$

$F(x,y) = (x, -y)$. $\quad \bar{F} = x+y + i(x-y)$

$$\frac{\partial F_1(x,y)}{\partial x} = 1, \quad \frac{\partial F_1(x,y)}{\partial y} = 0.$$

$$\frac{\partial F_2(x,y)}{\partial x} = 0, \quad \frac{\partial F_2(x,y)}{\partial y} = -1 \quad \Rightarrow \text{JM: } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Yet it's differentiable according to some thm.

Internezzo:

$F: \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at $(x_0, y_0) \in \mathbb{R}^2$ if there exists a function $\varphi: B_\varepsilon(x_0, y_0) \to \mathbb{R}^2$ s.t. $F((x_0+y_0)+h) = F(x_0,y_0) + \underbrace{DF(x_0,y_0)}_{\mathbb{R}^{2\times 2}} \underset{\mathbb{R}^2}{h} + \varphi((x_0,y_0)+h)$

$$\lim_{h\to 0} = \frac{\varphi((x_0,y_0)+h)}{\|h\|}$$

Jacobn Matrix: $DF(x_0, y_0) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1(x_0,y_0)}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_1(x_0,y_0)}{\partial y} \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ doesn't correspond to $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$

Thus, $f(z) = \bar{z}$ not differenable.

derivative of $R^n \to R^n$ function is a matrix.

Cauchy-Riemann Equations: (link two gradients)

1) Write $f: \mathbb{C} \to \mathbb{C}$ as $F: \mathbb{R}^2 \to \mathbb{R}^2$

2). Check F's differentiability

<u>Sufficient condition</u>: the four partial derivatives exist and continuous.

$\Rightarrow F$ is differentiable.

3). $DF = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} \in \mathbb{C}$ $\quad \frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y}, \quad \frac{\partial F_1}{\partial y} = \frac{-\partial F_2}{\partial x}$

$\partial F_1 \partial y = \partial F_2 \partial x, \; \partial F_1 \partial x = -\partial F_2 \partial y$

$\Rightarrow f$ is differentiable

THM: $f: \mathbb{C}$ to $\mathbb{C}$ is differentiable iff $F: \mathbb{R}^2 \to \mathbb{R}^2$ is diffentiable and Cauchy-Riemann equations hold.

<u>Proof</u>: $f(z) = F_1(x,y) + iF_2(x,y)$

$h \in \mathbb{R}$ $\quad \frac{f(z+h)-f(z)}{h} = \overbrace{\frac{F_1(x+h,y)-F_1(x,y)}{h}}^{\mathbb{R}} + \overbrace{i\,\frac{F_2(x+h,y)-F_2(x,y)}{h}}^{i\mathbb{R}}$

$\frac{f(z+ih)-f(z)}{ih} = \frac{F_2(x,y+h)-F_2(x,y)}{h} - i\,\frac{F_1(x,y+h)-F_1(x,y)}{h}$

Both LHS converge as $h \to 0$ to $f'(z)$

(1). $f'(z) = \frac{\partial F_1}{\partial x}(x,y) + i\frac{\partial F_2}{\partial x}(x,y)$

$f'(z) = \frac{\partial F_2(x,y)}{\partial y} - i\frac{\partial F_1}{\partial y}(x,y)$

$\Rightarrow$ CRE

F_1 is differentiable as $F_1((x_0,y_0)+h) = F_1(x_0,y_0)$

We have $f(z+h) = f(z) + f'(z)h + \varphi(z+h)$.

Observe: $f'(z) = \frac{\partial F_1(x,y)}{\partial x} - i\frac{\partial F_1}{\partial y}(x,y) = \frac{\partial F_2}{\partial y}(x,y) + i\frac{\partial F_2}{\partial x}(x,y)$.

$$F_1((x,y)+(h_1,h_2)) = F_1((x,y)) + DF_1(x,y)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \varphi_{F_1}((x,y)+(h_1,h_2))$$

$$Re(f(z+h)) = Re(f(z)) + Re(\underbrace{f'(z)}(h_1+ih_2)) + Re(\varphi(z+h))$$

$$= \underbrace{\frac{\partial F_1}{\partial x}(x,y) - i\frac{F_1}{\partial y}(x,y)}$$

$$= \frac{\partial F_1}{\partial x}(x,y)h_1 + \frac{\partial F_1}{\partial y}(x,y)h_2$$

$\varphi_{F_1}((x,y)+(h_1,h_2)) = Re(\varphi(z+h))$

$\Rightarrow \frac{\varphi_{F_1}((x,y)+(h_1,h_2))}{\|h\|} = \frac{|Re(\varphi(z+h))|}{|h|} \to 0$ as f differentiable.

$\Rightarrow F$ is differentiable, < do the same for F_2.

Converse:

$$f(z+h) = f(z) + \underbrace{f'(z)}_{} h + \varphi_z(h)$$

$$= \frac{\partial F_1}{\partial x} - i\frac{\partial F_1}{\partial y} \underset{\uparrow \text{CRE}}{=} \frac{\partial F_1}{\partial y} + i\frac{\partial F_2}{\partial x}$$

We want show $\frac{|\varphi_z(h)|}{|h|} \xrightarrow[h\to 0]{} 0$

We know. $F_1((x,y)+(h_1,h_2)) = \text{———} + \varphi_{F_1}((h_1,h_2))$

$Re(\varphi_z(h)) = \varphi_{F_1}(h_1,h_2) = 0$

$Im(\varphi_z(h)) = \varphi_{F_2}(h_1,h_2) = 0$

Thus from differentiability of $F_1, F_2 \Rightarrow \frac{|\varphi_z(h)|}{|h|} \xrightarrow[h\to 0]{} 0$

Wirtinger Calculus

$\partial_z = \frac{1}{2}(\partial x - i\partial y)$ \ define operator

$\partial_{\bar z} = \frac{1}{2}(\partial x + i\partial y)$ /

$\partial_z z = \frac{1}{2}(\partial x - i\partial_y)(x+iy) = \frac{1}{2}(\partial_x x + \partial_y y) = \frac{1}{2}(1+1) = 1$

$\partial_{\bar z} \cdot \bar z = 1$ $\boxed{\text{Go Over Partial Derivatives}}$

$$\partial_{\bar z} f(z) = \partial_{\bar z} F_1 + \partial_{\bar z} F_2 = \frac{1}{2}\underbrace{(\partial_x F_1 - \partial_y F_2)}_{=0} + \frac{i}{2}\underbrace{(\partial_y F_1 + \partial_x F_2)}_{=0} \quad \text{(by CRE)}$$

$$f(z) = F_1 + iF_2$$

Conclusion:

CRE $\Leftrightarrow \partial_{\bar z} f(z) = 0$, $\partial_{\bar z}$ is Cauchy-Riemann Operator

Application:

$f(z) = z^2 \cdot \bar z$ $\quad$ $\partial_{\bar z} f(z) = z^2 \Rightarrow f$ is only differentiable at $z=0$.

$g(z) = e^{-z^2}$ $\quad$ $\Rightarrow g$ is differentiable everywhere.

Do we need to consider $F: \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable?

$$\partial_{\bar z}\partial_z = \frac{1}{4}(\partial_x + i\partial_y)(\partial_x - i\partial_y) = \frac{1}{4}(\partial_x^2 + \partial_y^2) = \frac{1}{4}\Delta \ \boxed{!}$$

$\partial_x\partial_y = \partial_y\partial_x$ (Schwarz Lemma)

$$\Delta f = 4\partial_{\bar z}\partial_z f = 4\partial_z \underbrace{\partial_{\bar z} f}_{=0 \text{ CRE}} = 0 \quad [\text{harmonic}]$$

infinite timesly differentiable

Functions with $\Delta f = 0$ are called harmonic,

Its $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ also harmonic. $\Delta f = \Delta \mathrm{Re}(f) + \Delta \mathrm{Im}(f)$.

Def. An open and connected subset of $\mathbb{C}$ is called a domain.

Corollary: Let $D \subseteq \mathbb{C}$ be a domain and $f: D \to \mathbb{C}$ is complex differentiable. If f actually maps into $\mathbb{R}$, then f is constant.

Proof. $\because$ Maps into $\mathbb{R}$ $\therefore F_2 = 0$

$$\Rightarrow \left.\begin{matrix} \partial_x F_1 = \partial_y F_2 = 0 \\ \partial_y F_1 = -\partial_y F_2 = 0 \end{matrix}\right\} \Rightarrow \nabla F_1 = 0 \quad (*)$$

$$F_1(\underbrace{x_1, y_1}_{\vec{x}_1}) - F_1(\underbrace{x_0, y_0}_{\vec{x}_2}) = \int_0^1 \frac{d}{dt} F_1(X_0 + t(X_1 - X_0))\,dt$$

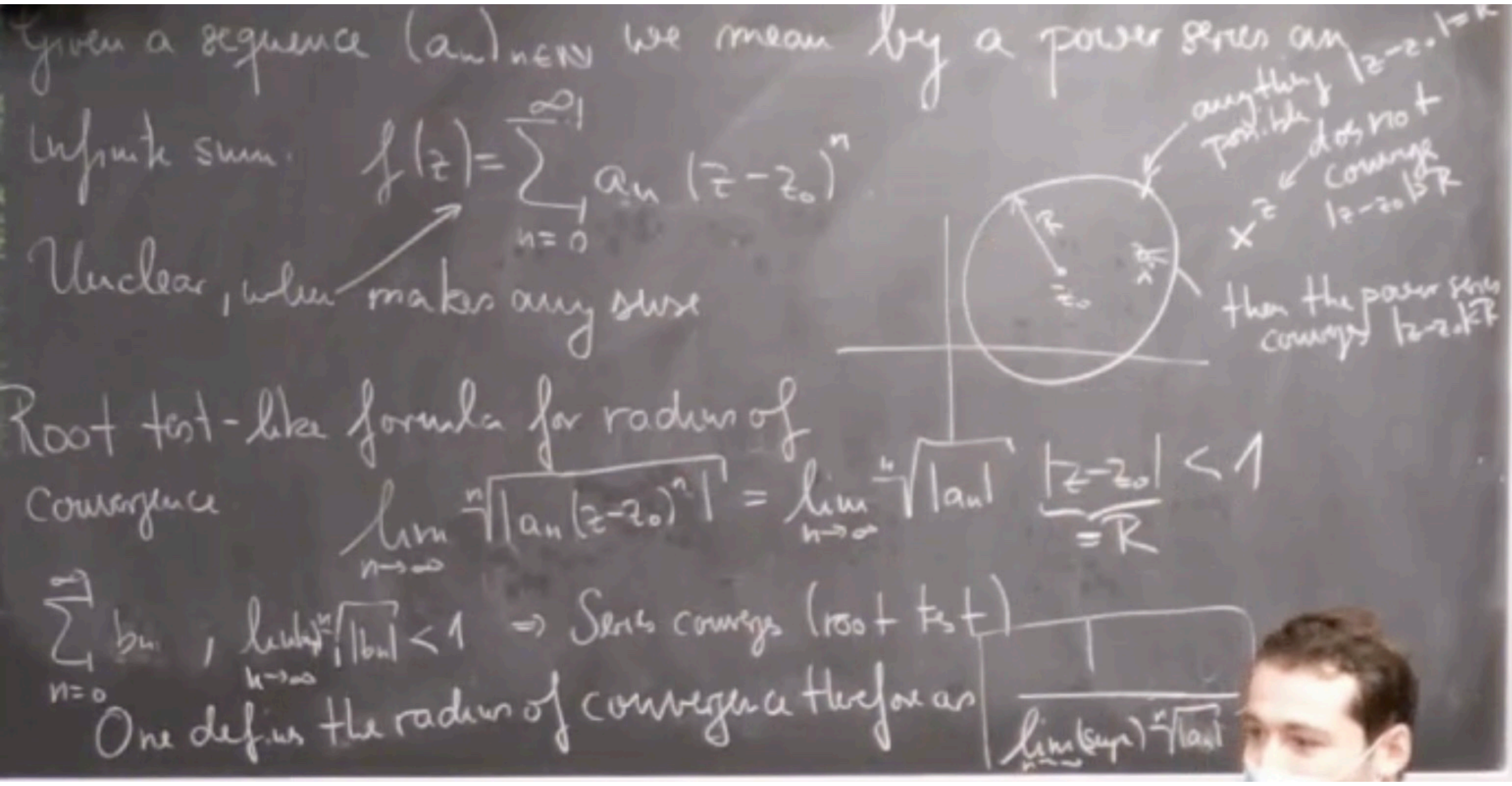
$$= \int_0^1 \underbrace{\nabla F_1(X_0 + t(X_1 - X_0))}_{=0 \ (*)} \cdot (X_1 - X_0)\,dt$$

$$= 0.$$

So, F_1 is constant.

f is constant as $F_1 \in C$ and $F_2 = 0$.

Power Series:



By root test. power series $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence $\frac{1}{\limsup_{n\to\infty}\sqrt[n]{|a_n|}}$.

(limsup indicates R always exists)

THM. Let $f\ B_R(z_0) \to \mathbb{C}$ be given by $f(z)=\sum_{n=1}^{\infty} a_n(z-z_0)^n$ with

$B_R(z_0) = \{z\in\mathbb{C} : |z-z_0|<R\}$

R being radius of convergence, then f is differentiable

(f differentiable within RoC).

Proof:

$$\frac{z^n-w^n}{z-w} = \underbrace{\sum_{k=0}^{n-1} z^{n-1-k} w^k}_{=(*)}, \quad z^n - w^n = (z-w)(*) \text{ is a telescopic sum}$$

$$\sum_{n=0}^{\infty} a_n \Bigg(\underbrace{\frac{z^n-w^n}{z-w}}_{=\sum_{k=0}^{n-1} z^{n-1-k} w^k}\Bigg) - g(w) = \sum_{n=0}^{\infty} a_n \Bigg(\underbrace{\sum_{k=0}^{n-1} z^{n-1-k} w^k - n\cdot w^{n-1}}\Bigg)$$

$$= \sum_{k=0}^{n-2} z^{n-k-1} w^k - (n-1) w^{n-1}$$

$$= \sum_{k=0}^{n-2} (k+1) z^{n-k-1} w^k - \underbrace{\sum_{k=0}^{n-2} k z^{n-k-1} w^k - (n-1) w^{n-1}}$$

$$= \sum_{k=0}^{n-1} k z^{n-k-1} w^k$$

$$= \sum^{n-1}$$

$$\sum a_n \Big(\qquad \Big) = (z-w) \sum_{n=0}^{\infty} a_n \sum_{k=1}^{n-1} k z^{n-k-1} w^{k-1} \xrightarrow[z \to w]{} 0$$

$\Rightarrow f$ is differentiable

Integration:

Def. $\displaystyle\int_{a\in\mathbb{R}}^{b\in\mathbb{R}} f(t)\,dt = \int_a^b \mathrm{Re}(f(t))\,dt + i\int_a^b \mathrm{Im}(f(t))\,dt$. $f: \mathbb{C}\to\mathbb{C}$

↑ maps to complex

Observation: The integral is linear, $\alpha,\beta\in\mathbb{R}$, f, g functions

$$\int_a^b \alpha f(t) + \beta g(t)\,dt = \alpha\int_a^b f(t)\,dt + \beta\int_a^b g(t)\,dt.$$

Triangular inequality: $\Big|\underbrace{\int_a^b f(t)\,dt}_{= r e^{i\varphi}}\Big| \leq \int_a^b |f(t)|\,dt$, Define $g(t) = f(t)e^{-i\varphi}$

Proof.

$$\int_a^b g(t)\,dt = e^{-i\varphi}\int_a^b f(t)\,dt = r\in\mathbb{R} \Rightarrow \int_a^b g(t)\,dt = \int_a^b \mathrm{Re}(g(t))\,dt$$

$$\Big|\int_a^b f(t)\,dt\Big| = \Big|\int_a^b g(t)\,dt\Big| = \Big|\int_a^b \mathrm{Re}(g(t))\,dt\Big|$$

z_1 $\quad$ z_2

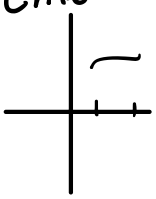
$$\leq \int_a^b \underbrace{|\mathrm{Re}(g(t))|}_{\leq |g(t)| = |f(t)|}\,dt$$

$z_2 = e^{-i\varphi} z_1$

$|z_2| = |z_1|$

$$\leq \int_a^b |f(t)|\,dt.$$

Line Integral: (Contour)



$$\int_\gamma f(z)\,dz$$

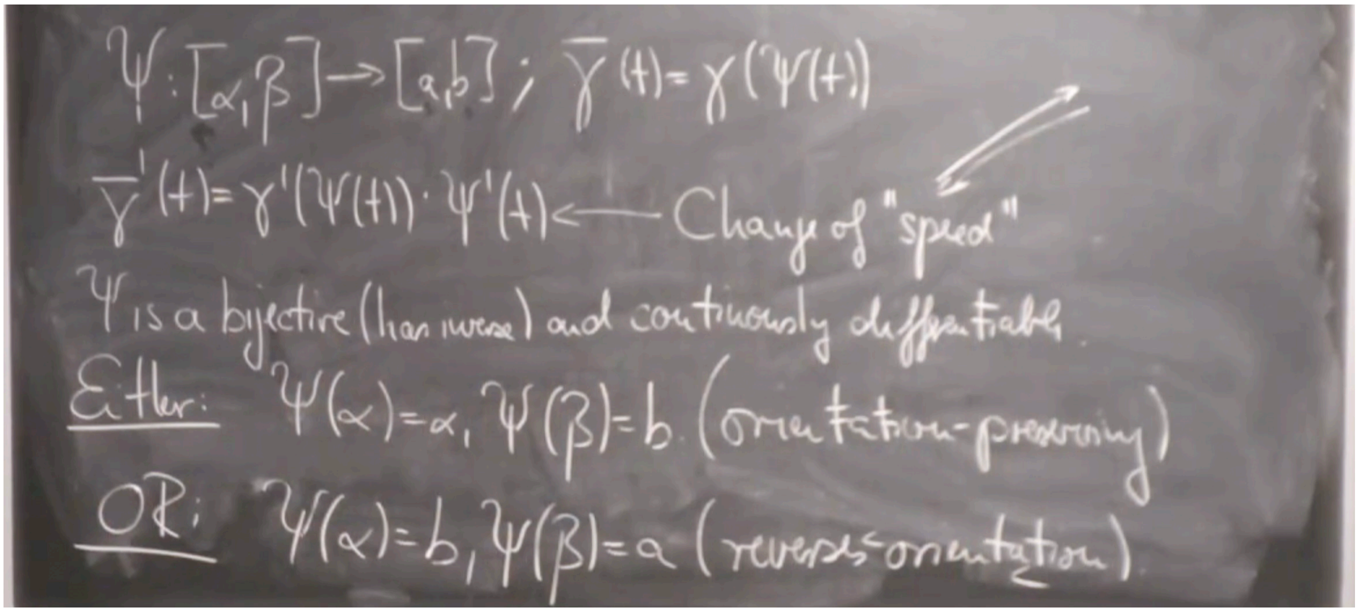
Def: Curve. A piece-wise continuously differentiable map $\gamma: [a,b] \to \mathbb{C}$ $\in\mathbb{R}$ is called a curve.

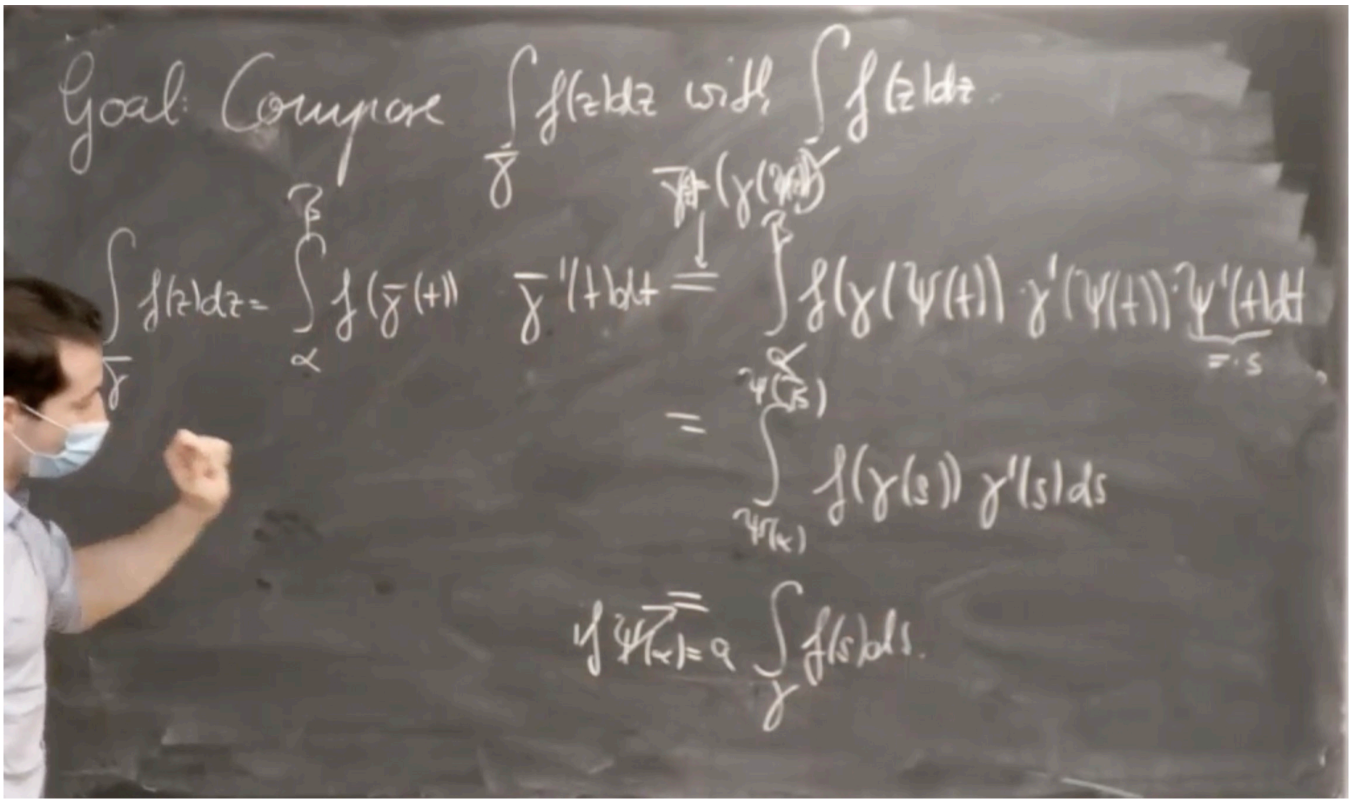
$$\int_a^b f(t)\,dt \simeq \sum_{i=1}^N f(t_i)(t_i - t_{i-1}) \ ; \ \sum_{i=1}^N f(\gamma(t_i))\big(\underbrace{\gamma(t_i) - \gamma(t_{i-1})}_{= \gamma'(t_i)(t_i - t_{i-1})}\big)$$

$$\simeq \sum_{i=1}^N f(\gamma(t_i))\gamma'(t_i)(t_i - t_{i-1})$$

Def. Let γ be a curve and $f: \mathbb{C}\to\mathbb{C}$ a continuous function.

$$\int f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt \ , \ \gamma: \mathbb{R}\to\mathbb{C}$$





Just a speed change won't change integral value.

$$\int_{\bar{\gamma}} f(z)\,dz = \pm \int_{\gamma} f(z)\,dz$$

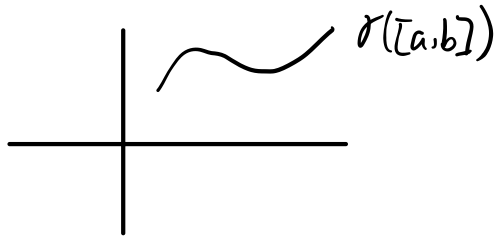
$$\boxed{\begin{aligned} \left|\int_{\gamma} f(z)dz\right| &\le \int_{\gamma} |f(z)||dz| \\ \| \\ \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| &\le \int_a^b |f(\gamma(t))||\gamma'(t)|dt \end{aligned}}$$

$$\gamma'(t) = \frac{d}{dt}(Re(\gamma(t))) + i\frac{d}{dt}(Im(\gamma(t)))$$

$$f'(z) = \frac{d}{dz}(Re(f(z))) + i\frac{d}{dz}(Im(f(z)))$$

Def. The length of a curve is defined as

$$\int_\gamma |dz| = \int_a^b |\gamma'(t)|\,dt$$

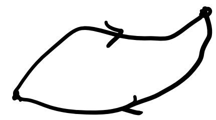


Triangular: If f is continuous, then

$$\left|\int_\gamma f(z)\,dz\right| \le \max_{z\in\gamma[a,b]} |f(z)| \int_\gamma |dz|$$

Proof. $\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt\right| \le \int_a^b \underbrace{|f(\gamma(t))|}_{a}\,|\gamma'(t)|\,dt$, $a \le \max_{z\in\gamma[a,b]} |f(z)|$

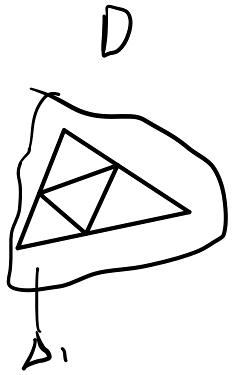
THM: Let $F: \mathbb{C}\to\mathbb{C}$ be complex differentiable with $F' = f$ and f is continuous, then for any closed curve γ $\int_\gamma f(z)\,dz = 0$



Proof:

$$\int_\gamma f(z)\,dz = \int_a^b \underbrace{f(\gamma(t))\,\gamma'(t)}_{= \frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\,\gamma'(t)}\,dt = \int_a^b \frac{d}{dt}F(\gamma(t))\,dt = F(\gamma(b)) - F(\gamma(a)) = 0$$

Ex. $f(z) = \frac{1}{z}$



$f: \mathbb{C}\to\mathbb{C}$ is complex differentiable (analytic) and $\Delta_1 \subseteq D$. Let γ be a closed curve on the boundary of Δ_1, then $\int_\gamma f(z)\,dz = 0$.

Proof. $\int_\gamma f(z)\,dz = \sum_{j=1}^{4} \int_{\gamma_j} f(z)\,dz$

$$\left|\int_\gamma f(z)\,dz\right| \le 4\left|\int_{\gamma_1} f(z)\,dz\right|$$

$\gamma_1 \leftarrow \max\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$

$L(\gamma_1) = \frac{1}{2} L(\gamma)$ # mid point

$\mathrm{diam}(\Delta_1) = \frac{1}{2}\mathrm{diam}(\Delta)$. $\mathrm{diam} = \max\{|x-y|, x, y \in M\}$

Repeat the process n many times. $\Delta \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_n$

$$\left|\int_\gamma f(z)\,dz\right| \leq 4^n \left|\int_{\gamma_n} f(z)\,dz\right|$$

$L(\gamma_n) = \frac{1}{2^n} L(\gamma)$, $\mathrm{diam}(\Delta_n) = \frac{1}{2^n}\mathrm{diam}(\Delta)$

$\bigcap_{n\in\mathbb{N}} \Delta_n = \{z_0\}$. BW- theorem

$$\int_{\gamma_n} f(z)\,dz = \int_{\gamma_n} f(z) - f(z_0) - f'(z_0)(z-z_0)\,dz + \underbrace{\int_{\gamma_n} f(z_0) + f'(z_0)(z-z_0)\,dz}_{=0}$$

$F(z) = f(z_0)\cdot z + \frac{1}{2} f'(z_0)(z-z_0)^2$. $F'(z) = f(z_0) + f'(z_0)(z-z_0)$

$$\therefore \int_{\gamma_n} f(z)\,dz = \int_{\gamma_n} \underbrace{f(z) - f(z_0) - f'(z_0)(z-z_0)}_{=\varphi_{z_0}(z)}\,dz,$$ Remember f is differentiable

$$\lim_{z\to z_0} \frac{\varphi_{z_0}(z)}{|z-z_0|} = 0 \Rightarrow |\varphi_{z_0}(z)| \leq \varepsilon|z-z_0| \text{ for } \forall \varepsilon > 0 \text{ if } z \in B_{r(\varepsilon)}(z_0)$$

$$\left|\int_\gamma f(z)\,dz\right| \leq 4^n \left|\int_{\gamma_n} f(z)\,dz\right| \leq 4^n \int_{\gamma_n} |\varphi_{z_0}(z)|\,|dz| \leq 4^n \varepsilon \int_{\gamma_n} \underbrace{|z-z_0|}_{\leq \mathrm{diam}(\Delta_n) \leq \frac{\mathrm{diam}(\Delta)}{2^n}}\,|dz|$$

$$\leq 2^n \varepsilon \int_{\gamma_n} |dz| \leq 2^n \varepsilon \frac{1}{2^n} \int_\gamma |dz| \leq \varepsilon \int_\gamma |dz| \quad \blacksquare$$

A set $S \subseteq \mathbb{C}$ is called convex $\forall x, y \in S$, $\forall t \in [0,1]$, $x + t(y-x) \in S$

THM. Let $D \subseteq \mathbb{C}$ be a domain that is also convex and $f: D \to \mathbb{C}$ analytic. Let γ be a closed curve in D, then $\int_\gamma f(z)\,dz = 0$.

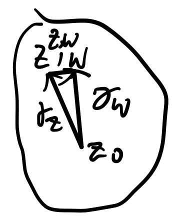
Proof. Goal: Find $F: D \subseteq \mathbb{C} \to \mathbb{C}$ s.t. $F'(z) = f(z)$

Remark: If $\mathbb{R}$ instead of $\mathbb{C}$. Take arbitrary $x_0 \in \mathbb{R}$:
$F(x) = \int_{x_0}^{x} f(t)\,dt$ then FTOC: $F'(x) = f(x)$

Same idea: Fix $z_0 \in D$. Define $\gamma_z(t) = z_0 + t(z - z_0)$

Observe that by convexity: $\gamma_z(t) \in D$ for all $t \in [0,1]$

Define $F(z) = \int_{\gamma_z} f(w)\,dw$. [Show $F'(z) = f(z)$]



$$F(z) - F(w) = \int_{\gamma_z} f(s)\,ds - \int_{\gamma_w} f(s)\,ds$$

$$= \underbrace{\int_{\gamma_z} f(s)\,ds + \int_{-\gamma_w} f(s)\,ds + \int_{\gamma_{z,w}} f(s)\,ds} - \int_{\gamma_{z,w}} f(s)\,ds$$

$$= \int_{\triangle} f(s)\,ds - \int_{\gamma_{z,w}} f(s)\,ds$$

$$= -\int_{\gamma_{z,w}} f(s)\,ds$$

$$\frac{F(z) - F(w)}{z - w} = -\int_{\gamma_{z,w}} f(s)\,ds \,\frac{1}{z-w} = \int_0^1 f(z + t(w-z)) \frac{(w-z)}{z-w}\,dt \quad , \quad \gamma_{z,w}(t) = z + t(w-z)$$

$$= \int_0^1 f(z + t(w-z))\,dt$$

$$\lim_{w\to z}\left(\frac{F(z)-F(w)}{z-w}-f(z)\right)=\lim_{w\to z}\int_0^1 \underline{f(z+t(w-z))-f(z)}\,dt=0. \quad \text{☐}$$

Since f is continuous at z.

$\forall \varepsilon>0\ \exists \delta>0\ \forall w\in\mathbb{C}: |z-w|<\delta \Rightarrow |f(z)-f(w)|<\varepsilon$

Idea: Value of line integral in complex analysis are topological invariants.

Homotopy: Let $\gamma_1,\gamma_2:[0,1]\to\mathbb{C}$ closed curve, then a homotopy H is a continuous map. $H\ [0,1]\times[0,1]\to\mathbb{C}$ with the following properties (Continuously transformed to each other)

$H(0,t)=\gamma_1(t)$ (time parameter of the curve) $t\in[0,1]$

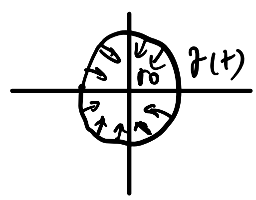
$H(1,t)=\gamma_2(t)$

$H(s,0)=H(s,1)$ for all $s\in[0,1]$ We always have a closed curve while transforming them.

Def. A curve is called null-homotopic if it is homotopic to a constant curve, i.e. a curve $\gamma(t)=z$ for all $t\in[0,1]$ where $z\in\mathbb{C}$ is a fixed point.

E.X. $\gamma(t)=e^{2\pi i t}$; $D=\{|z|\leq 1\}$

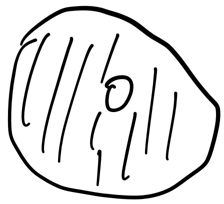
$\gamma_0(t)=0$.



$H(s,t)=(1-s)e^{2\pi i t} \to H(0,t)=\gamma(t)$ $\quad H(s,0)=H(s,1)$

$H(1,t)=0=\gamma_0(t)$

Def. A set $S \subseteq \mathbb{C}$ where every closed curve is null-homotopic is called simply connected.



$\leftarrow$ connected but not simply connected.

But every convex set is simply connected.

Compactness: A set $K \subseteq \mathbb{C}$ is called compact if any of the following equivalent definition hold:

(i) K is closed and bounded.

(ii) K has Bolzano-Weistrass property

Every sequence $(z_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Remark: $f: D \to \mathbb{C}$ is called continuous at $z_0 \in D$ if $\forall \varepsilon > 0\ \exists \underset{\delta(\varepsilon, z_0)}{\delta > 0}\ \forall z \in D\ |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$.

Uniform: $\forall \varepsilon > 0\ \exists \delta\ \forall z, w \in D: |z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon$

Lemma: If $f: K \to \mathbb{C}$ is continuous and K is compact, then f is uniformly continuous.

Proof. $\exists \varepsilon > 0$, $\forall \frac{1}{n}, n \in \mathbb{N}, \exists z_n, w_n: |z_n - w_n| < \frac{1}{n}$ and $|f(z_n) - f(w_n)| > \varepsilon$ (Proof by contradiction)

z_n has a convergent subsequence: $z_{n_k} \to z$

$\Rightarrow |w_n - z| \leq \underbrace{|z_n - z|}_{\to 0} + \underbrace{|z_n - w_n|}_{\to 0} \Rightarrow w_n \to z$

f is continuous: $z_n \to z \Rightarrow f(z_n) \to f(z)$

$w_n \to z \Rightarrow f(w_n) \to f(z)$ $\Big\} (\Rightarrow\Leftarrow)$ ▨

Let $f: K \to \mathbb{R}$ be continuous and K is compact, then f attains min and maximum.

$\Leftrightarrow \exists z \in K.\ f(z) = \sup\limits_{w \in K} f(w)$, $f(z) = \inf\limits_{w \in K} f(w)$.

Proof. $\exists w_n \in K$, $\lim\limits_{n\to\infty} f(w_n) = \sup\limits_{w \in K} f(w)$.

$\exists$ subsequence $w_{n_k} \to w \Rightarrow f(w_{n_k}) \to f(w) = \sup\limits_{w \in K} f(w)$

THM (Cauchy): $f: D \to \mathbb{C}$ analytic. Let γ_0, γ_1 be two closed homotopic curves in D, then $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$

If γ_0 is null-homotopic, then $\int_{\gamma_0} f(z)\,dz = 0$.

If D simply connected. every closed γ integrates to 0.

Lemma: Let $f: K \to \mathbb{C}$ be continuous and K is compact, then $f(K)$ is compact.

Proof. Let $f(w_n)$ be a sequence in $f(K)$. $w_n \in K$.

Since K is compact, w_n has a cgt subsequence, so $w_{n_k} \to w \in K$. $f(w_{n_k}) \to f(w)$. ▨

Proof. (Cauchy).

$H:\underbrace{[0,1]\times[0,1]}_{\text{compact.}} \to D$, CTS. $\Rightarrow$ uniformly CTS (THM)

So, $\underbrace{H([0,1]\times[0,1])} = \{z\in D : H(s,t)=z \text{ for some } s,t\}$.

$= \underset{\subseteq D}{K}$ (compact by previous lemma).

Observe: $\varphi(z) = d(z, \mathbb{C}\setminus D)$ is a CTS function.

$$= \begin{cases} 0, & \text{if } z\in \mathbb{C}\setminus D \\ \inf_{w\in\mathbb{C}\setminus D}|w-z|, & \text{if } z\in D \end{cases}$$

$\varphi|_K$ attains minimum.

$\Rightarrow \inf_{w\in K}\varphi(w) \geq \varepsilon > 0$ for some $\varepsilon>0$.

Since H is uniformly continuous, we can choose $m\in\mathbb{N}$, $\frac{1}{m}<\delta$.

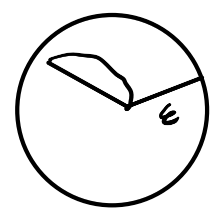
$|s-s'|<\frac{1}{m}$, $|t-t'|<\frac{1}{m} \Rightarrow |H(s,t)-H(s',t')|<\varepsilon$.

Define $\pi_{\frac{k}{m}}$ by discreptizing $H(\frac{k}{m},\frac{k'}{m})$, $k'\in\{0,\dots,m\}$

1). Define m many curves and observe they're fully in D.

2). $\int_{\gamma_1} f(z)\,dz = \int_{\pi_0} f(z)\,dz$ and $\int_{\gamma_2} f(z)\,dz = \int_{\pi_1} f(z)\,dz$.

3). $\int_{\pi_{\frac{k}{m}}} f(z)\,dz = \int_{\pi_{\frac{k+1}{m}}} f(z)\,dz$ for $k\in\{0,\dots,m\}$.

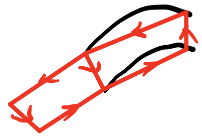


$\int_{\gamma_1|_{[0,\frac{1}{m}]}} f(z)\,dz = \int_{\pi_0|_{[0,\frac{1}{m}]}} f(z)\,dz \Rightarrow \int_{\gamma_1} f(z)\,dz = \int_{\pi_0} f(z)\,dz$.

ball is convex

$\int_{\gamma_2} f(z)\,dz = \int_{\pi_1} f(z)\,dz$

$$\sum \int_{red} f(z)\,dz = 0. \Rightarrow \int_{\pi\frac{k}{m}} f(z)\,dz = \int_{\pi\frac{k+1}{m}} f(z)\,dz.$$



四

Theorem: Let $f: D \to \mathbb{C}$ analytic, assume that γ_r is inside D. So for all z s.t. $|z - z_0| < r$

$$\boxed{\gamma_r(t) = z_0 + re^{2\pi i t}}$$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w-z}\,dw$$ (Knowing f on the boundary define f everywhere inside)

$+$ (differentiable ∞ times)

Proof. By Cauchy's integral theorem,

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{w-z}\,dw, \text{ where } \gamma_R = z + Re^{2\pi i t}$$

Since f is differentiable, f has to be CTS, therefore

$\forall \varepsilon > 0, \exists \delta(z,\varepsilon)\ \forall w \in \mathbb{C}$

$$\left| \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{w-z}\,dw - f(z) \right| = \left| \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{w-z} - \frac{f(z)}{w-z}\,dw \right|$$

$$\underset{\text{triangular}}{\underset{\uparrow}{\leq}} \frac{1}{2\pi i} \int_{\gamma_R} \frac{|f(w) - f(z)|}{|w-z|}\,d|w|$$

$$\underset{R<\delta}{\underset{\uparrow}{\leq}} \varepsilon \int_{\gamma_R} \frac{1}{|w-z|}$$

$$= \varepsilon \int_0^1 \frac{1}{|Re^{2\pi i t}|} |\gamma_R'(t)|\,dt = \varepsilon.$$

$$\underbrace{}= |Re^{2\pi i t} 2\pi i| = 2\pi R$$

Def. Let $K \subseteq \mathbb{C}$ and $g_n : K \to \mathbb{C}$ and $g : K \to \mathbb{C}$. then g_n is called uniformly convergent to g if $\lim_{n\to\infty} \sup_{z\in K} |g_n(z) - g(z)| = 0$

E.X. $K = [-1,1]$, $g_n(x) = \sqrt{x^2 + \frac{1}{n}}$, $g(x) = |x|$

$$g_n(x) - g(x) = \sqrt{x^2+\frac{1}{n}} - \sqrt{x^2}$$

$$= \frac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}} + \sqrt{x^2}}$$

$$\leq \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \frac{1}{\sqrt{n}}$$

$|g_n(x) - g(x)| \leq \frac{1}{\sqrt{n}}$. $\sup_{x\in[-1,1]} |g_n(x) - g(x)| \leq \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \sup_{x\in[-1,1]} |g_n(x) - g(x)| \leq \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Lemma: Let $\gamma:[0,1]\to K$ be a curve and $g_n: K\to \mathbb{C}$,

$g: K\to\mathbb{C}$ s.t. g_n converges uniformly to g.

then $\int\limits_\gamma \lim\limits_{n\to\infty} g_n(z)\,dz = \int\limits_\gamma \underbrace{g(z)}_{=\lim\limits_{n\to\infty} g_n(z)}\,dz$

In addition, if g_n are continuous, then also $\lim g_n$ is continuous.

Uniform convergent preserves continuity.

E.X. $g_n:[0,1]\to\mathbb{C}$, $g_n = x^n$

for every fixed x:

$$\lim_{n\to\infty} g_n(x) = \begin{cases} 0, & \text{if } x<1 \\ 1, & \text{if } x=1 \end{cases}$$

Proof.

$$\left|\int_\gamma g_n(z)dz - \int_\gamma g(z)dz\right| = \left|\int_\gamma g_n(z) - g(z)\,dz\right|$$

$$\underset{\Delta\text{-inequal.}}{\leq} \int_\gamma \sup_{z\in\gamma([0,1])} |g_n(z)-g(z)|\,|dz|$$

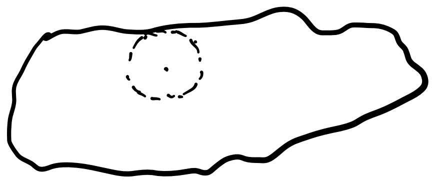
$$= \underbrace{\sup_{z\in\gamma([0,1])} |g_n(z)-g(z)|\,\text{length}(\gamma)}_{\xrightarrow[n\to\infty]{} 0}$$

Theorem: Let $f: D \to \mathbb{C}$ be analytic. Then f can be written in terms of a power series for every $a \in D$.

$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ and in particular f is infinitely many differentiable

The power series converges for every z such that z is inside a disk fully contained in D.

In particular, $a_n = \frac{f^{(n)}(a)}{n!}$



$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw$$

$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}, \quad \sum z^n = \frac{1}{1-z}$$

$$= \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n$$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n dw$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw}_{a_n} (z-a)^n$$

uniform converge

Claim: $g_m(z) = \sum_{n=0}^{m} z^n$ converges uniformly to $g(z) = \sum_{n=0}^{\infty} z^n$ for $|z| \le r < 1$

$$\sup_{|z| \le r < 1} \left| \sum_{n=0}^{m} z^n - \sum_{n=0}^{\infty} z^n \right| = \sup_{|z| \le r < 1} \left| \sum_{n=m+1}^{\infty} z^n \right| \le \sum_{n=m+1}^{\infty} r^n \xrightarrow[m \to \infty]{} 0$$

Corollary: For f in the theorem, we have $\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw$

Application: $\int_{|w|=r} \frac{\overbrace{\sin(e^w)}^{=f(w)}}{w^2}\, dw = 2\pi i f'(0)$

Liouville's Theorem | Let $f: \mathbb{C} \to \mathbb{C}$, analytic (holomorphic) (complex differentiable everywhere) called entire. If f is bounded ($\sup_{z \in \mathbb{C}} |f(z)| < \infty$). then f is constant

$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$ $\qquad$ $f(z) = \sin(z)$ is unbounded.

$x \in \mathbb{R}$ $\quad \sin(ix) = \frac{1}{2i}(e^{-x} - ie^{x})$ along imaginary line, goes to inf.

Proof. $f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|w|=r} \frac{f(w)}{(w-z)^{n+1}} dw$

$$|f^{(n)}(z)| \le \frac{n!}{2\pi} \left| \int_{|w|=r} \frac{f(w)}{(w-z)^{n+1}} dw \right| \le \frac{n!}{2\pi} \sup_{|w|=r} \frac{|f(w)|}{|w-z|^{n+1}} 2\pi r$$

$$= r\, n! \sup_{|w|=r} \frac{|f(w)|}{|w-z|^{n+1}}$$

$\Rightarrow |f^{(n)}(0)| \le \frac{n!}{r^n} \sup_{|w|=r} |f(w)| \xrightarrow[r \to \infty]{} 0$ for $n \ge 1$, since f is bounded.

$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f(0)$. constant

Fundamental Theorem of Algebra: !

Theorem: Every non-constant polynomial has a root in $\mathbb{C}$.

Proof. $p(z) = a_0 + a_1 z + \cdots + a_n z^n \leftarrow$ entire. Assume $p(z) \neq 0$ for $\forall z \in \mathbb{C}$

$q(z) = \frac{1}{p(z)} \leftarrow$ is also entire. q is also bounded since

$\lim_{|z| \to \infty} p(z) = \infty \Rightarrow \lim_{|z| \to \infty} \frac{1}{p(z)} = 0$. Fix $\varepsilon > 0$, then $\exists R > 0$. $|z| \ge R$, $\frac{1}{|q(z)|} < \varepsilon$.

Observe that $|z| \le R$ is a compact set.

Recall $q(z) = \frac{1}{p(z)}$ is CTS and thus $|q(z)| \le r'$ for $\forall |r| \le R$. $\Rightarrow q$ is constant. (by Liouville's Theorem)

Theorem: Let $f: D \to \mathbb{C}$ be analytic and D is simply connected.
Then f has an antiderivative $F: D \to \mathbb{C}$ s.t. $F' = f$.
$f(z) = z$, $F(z) = \frac{z^2}{2} + C$

Fix $z_0 \in D$, then take any curve $\gamma [0,1] \to D$ s.t. $\gamma(0) = z_0$, $\gamma(1) = z$.
$F(z) = \int_\gamma f(w) dw$.

curve independent. $x = y \Rightarrow f(x) = f(y)$.

Proof. The F is well-defined. let $\gamma_1, \gamma_2 : [0,1] \to D$ be two curves starting at z_0. $\gamma_1(0) = \gamma_2(0) = z_0$ and connecting it to z: $\gamma_1(1) = \gamma_2(1) = z$.
In order for F to be well-defined $\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$,
since $\int_{\gamma_1} f(z) dz + \int_{-\gamma_2} f(z) dz = 0$.

$$f(x) = \begin{cases} e^{\frac{1}{1-x}}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$ infinitely differentiable.

Theorem: Let f, g $D \to \mathbb{C}$ be analytic, then the following are equivalent

(i). $f = g$
(ii) $\exists z_0 \in D$. $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \geq 0$.
(iii). $\exists z_0 \in D$ and sequence $z_n \to z_0$ $f(z_n) = g(z_n)$

Proof (i)$\Rightarrow$(ii), (i)$\Rightarrow$(iii)

$h = f - g$. $\Rightarrow$ If $h^{(n)}(z_0) = 0$ for all $n \geq 0$. $h(z) = \sum_{n=0}^{\infty} \frac{\overbrace{h^{(n)}(z_0)}^{=0}}{n!} (z - z_0)^n = 0$.

This shows (ii)$\Rightarrow$(i).

Now (iii) $\Rightarrow$ (i).
$h(z_n) = 0$ and since $z_n \to z_0 \Rightarrow h(z_n) \to h(z_0) = 0$.

$h(z) = \sum_{n=m} \frac{h^{(n)}(z_0)}{n!}(z-z_0)^n$, $m \geq 1$. We assume m is such that $h^{(m)}(z_0) \neq 0$

$$h(z) = \underbrace{(z-z_0)^m}_{\neq 0} \underbrace{\sum_{n=0}^{\infty} \frac{h^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n}_{g(z_n)=0.}$$

$h(z_n) = 0$ $\quad$ $z_n \to z_0 \Rightarrow g(z_n) \to g(z_0) = 0.$

$$0 = g(z_0) = \frac{h^{(m)}(z_0)}{m!}$$

Maximum Principle:

Let $f: D \to \mathbb{C}$ be analytic s.t. exists $z \in D$ $|w-z| < \varepsilon \Rightarrow |f(w)| \leq |f(z)|$. $\Rightarrow f$ is constant.

Proof. $|f(z)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2 dt$. $\quad |f(z)| \geq |f(z+re^{it})|$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z+re^{it})|^2}_{f(z+re^{it})\overline{f(z+re^{it})}} dt \qquad f(z+re^{it}) = \sum_{n=0}^{\infty} c_n (re^{it})^n \leftarrow \text{series.}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{\infty} c_n r^n e^{itn} \sum_{m=0}^{\infty} \overline{c_m} r^m e^{-itm} dt$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n \overline{c_m} \frac{r^{m+n}}{2\pi} \underbrace{\int_0^{2\pi} e^{it(n-m)} dt}_{\text{can be compute with sin/cos}} = \begin{cases} 1, & n=m \\ 0, & n \neq m \end{cases}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n \overline{c_m} r^{n+m} \delta_{n,m} \quad \leftarrow \text{Kroucher Delta.} \quad \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j, \end{cases}$$

$$= \sum_{n=0}^{\infty} \underbrace{|c_n|^2}_{\left(\frac{|f^{(n)}(z)|}{n!}\right)^2} r^{2n} = |f(z)|^2 + \frac{|f^{n}(z)|^2}{1} r^2 + \cdots$$

$\Uparrow$

lower bound

$f^n(z) = 0$ for all $n \in \mathbb{N}$. $\Rightarrow f$ is constant $\leftarrow$ ☐

Corrollary:

Let $f: D \to \mathbb{C}$ analytic. D is bounded and $f: \bar{D} \to \mathbb{C}$ ~~ver~~ CTS. Then

$$\max_{z \in D} |f(z)| = \max_{z \in \partial D} |f(z)|$$

Proof. Trivial.

Theorem:

Let $f: D \to \mathbb{C}$ continuous. $\int_{\Delta} f(z)\,dz = 0$. for all $\Delta \subseteq D$.

$\Rightarrow f$ is analytic

Differentiability not preserved under uniform convergen

But complex diff... is!

Weierstrass convergence THM: If $\{f_n\}$. $f_n: D \to \mathbb{C}$ analytic and converge uniformly to $f: D \to \mathbb{C}$. then f is also analytic.

Proof. $\int_{\Delta} f(z)\,dz = \lim_{n \to \infty} \int_{\Delta} f_n(z)\,dz = 0$

uniform convergence.

Def. Let $\gamma[0,1]\to\mathbb{C}$ be closed curve and $z\notin\gamma([0,1])$ then the winding number $n(\gamma,z)=\frac{1}{2\pi i}\int_\gamma\frac{dw}{w-z}$

Example: $\gamma(t)=e^{2\pi int}$, $t\in[0,1]$, $n\in\mathbb{Z}$

$$n(\gamma,0)=\frac{1}{2\pi i}\int_0^1\frac{1}{\gamma(t)}\gamma'(t)\,dt=n$$

a is an integer iff $e^{2\pi ia}=1$.

Lemma: The winding number is always an integer and is constant on connected components.

$|z-z'|<\delta(\varepsilon)\Rightarrow|n(\gamma,z)-n(\gamma,z')|<\varepsilon$. (continuity)

Theorem: Let $\gamma:[0,1]\to D$ a closed curve and $f:D\to\mathbb{C}$ is analytic, then

$$f(z)n(\gamma,z)=\frac{1}{2\pi i}\int_\gamma\frac{f(w)}{w-z}dw\ ,\quad z\notin\gamma([0,1])$$

Square / Powers / Log.

Def. Let D be a domain then we call a function $g: D \to \mathbb{C}$ a log if it has
i) g is analytic on D.
ii). $e^{g(z)} = z$, $\forall z \in D$.

Question: Is it possible that $0 \in D$? Maybe $D = \mathbb{C} \setminus \{0\}$

No, $e^{g(z)} \cdot g'(z) = (z)' = 1 \Rightarrow g'(z) = \frac{1}{z}$.

$\int_\gamma \frac{1}{z} dz = 0$ by theorem. But this case $\neq 0$, $\gamma(t) = e^{2\pi i t}$.

Theorem: Let D be simply connected domain s.t. $0 \notin D$, then $\exists$ $g: D \to \mathbb{C}$ that satisfies the definition of a log up to a multiple of $2\pi i$.

$z^\alpha = e^{\alpha \log(z)}$

Corollary: $z^\alpha = e^{\alpha g(z)}$ is well-defined analytic for $z \in D \setminus \{0\}$, where D is any simple connected domain.

Poles and Singularities:

$\swarrow \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$

1). Removable Singularity : $f(z) = \frac{\sin z}{z}$, $z \in \mathbb{C} \setminus \{0\}$.

2). Pole: $f(z) = \frac{1}{z}$, $z \in \mathbb{C} \setminus \{0\}$.

3). Essential Singularity: $f(z) = e^{\frac{1}{z}}$, $z \in \mathbb{C} \setminus \{0\}$.

$f(\frac{1}{n}) = e^n \xrightarrow[n\to\infty]{} \infty$, not ①. $f(-\frac{1}{n}) = e^{-n} \xrightarrow[n\to\infty]{} 0$ Not pole.

Def ① Let $z_0 \in D$ and $f: D \setminus \{z_0\} \to \mathbb{C}$ is analytic. Then if there exists an analytic function $\tilde{f}: D \to \mathbb{C}$ with $\tilde{f}|_{D\setminus\{0\}} = f$, then z_0 is called a removable singularity.

② Let $z_0 \in D$ and $f: D \setminus \{z_0\} \to \mathbb{C}$ is analytic, then f has a pole if $\lim_{z \to z_0} |f(z)| = \infty$

③ A function which has only removable singularity and poles is called Memorphic.

④ If $z_0 \in D$ and $f: D \setminus \{z_0\} \to \mathbb{C}$ analytic has neither a pole nor a removenable singularity at z_0, then z_0 is called an essential singulerity.

Theorem (Removable Singularities, Riemann).

$z_0 \in D$. $f: D\setminus\{z_0\} \to \mathbb{C}$ is analytic, then z_0 is removable singularity iff f is bounded in a neighborhood of z_0.

Theorem (Essential Singularities)

Let $z_0 \in D$, and $f: D\setminus\{z_0\} \to \mathbb{C}$ analytic with an essential singularity at z_0. Then, for any $\delta > 0$: $f(B_\delta(z_0)\setminus\{z_0\})$ is dense in $\mathbb{C}$.

Theorem (Poles)

$z_0 \in D$ and $f: D\setminus\{z_0\} \to \mathbb{C}$ has a pole, then there's a natural number $m \in \mathbb{N}$ and an analytic function $g: D \to \mathbb{C}$ with $g(z_0) \neq 0$ such that $f(z) = \dfrac{g(z)}{(z-z_0)^m}$. the pole is of order m.

(All analytic f with sing has such facto)

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^{n-m} = \sum_{n=-m}^{\infty} c_{n+m}(z-z_0)^n \leftarrow \text{Laurent Series}$$

<u>Remark</u>: If f has an essential sing, then $f(z) = \sum_{-\infty}^{\infty} c_n (z-z_0)^n$

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{z^n \cdot n!} = \sum_{n=-\infty}^{0} \frac{z^n}{|n|!}$$

Residue Theorem:

Let γ be a closed null-homotopic curve inside domain D. Let $f: D \to \mathbb{C}$ be meromorphic. Let $z_1 \cdots z_n \in D$ be the pole of f inside D enclosed by γ, then

$$\frac{1}{2\pi i} \oint_\gamma f(z)\, dz = \sum_{n=1}^{N} n(\gamma, z_n)\, \operatorname{res}(f, z_n).$$

The residue is the coefficient of C_{-1} in the Laurent Series.

$$f(z) = \sum_{n=m}^{\infty} c_n (z - z_k)^n \Rightarrow \operatorname{res}(f, z_k) = C_{-1}, \quad m = \text{order of pole at } z_k.$$

Ex. $\displaystyle \frac{1}{2\pi i} \oint_\gamma \frac{1}{z}\, dz = \underbrace{n(0,\gamma)}_{=1} \underbrace{\operatorname{res}\left(\frac{1}{z}, 0\right)}_{=1} = 1 \quad \gamma(t) = e^{2\pi i t}.$

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{z^3}\, dz = n(\gamma, 0)\, \operatorname{res}\left(\frac{1}{z^3}, 0\right) = 0 \quad \text{since } C_{-1} = 0.$$

$$\frac{1}{z} = \sum_{n=-1}^{\infty} c_n z^n, \quad c_n = 0,\ n \geq 0,\ C_{-1} = 1 = \operatorname{res}\left(\frac{1}{z}, 0\right).$$

Theorem: Let f be a meromorphic function.

$f: D \to \mathbb{C}$, $\gamma [0,1] \to D$ is a closed curve.

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \# \text{ of zero}(f) - \# \text{ of poles}(f)$$

(order counts)

Remark: $\frac{d}{dz} \log(f(z)) = \frac{f'(z)}{f(z)}$

Theorem: Let $f: D \to \mathbb{C}$ be analytic. Then f can be written in terms of a power series for every $a \in D$.

$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ and in particular f is infinitely many differentiable

The power series converges for every z such that z is inside a disk fully contained in D.

In particular, $a_n = \frac{f^{(n)}(a)}{n!}$

$$f(z) = \frac{1}{2\pi} \int_{\gamma} \frac{f(w)}{w-z} \, dw$$

$$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)(1-\frac{z-a}{w-a})} \quad , \sum z^n = \frac{1}{1-z}$$

$$= \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n$$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n dw$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \, dw}_{a_n} (z-a)^n$$

uniform converge

Claim: $g_m(z) = \sum_{n=0}^{m} z^n$ converges uniformly to $g(z) = \sum_{n=0}^{\infty} z^n$ for $|z| \leq r < 1$

$$\sup_{|z| \leq r < 1} \left| \sum_{n=0}^{m} z^n - \sum_{n=0}^{\infty} z^n \right| = \sup_{|z| \leq r < 1} \left| \sum_{n=m+1}^{\infty} z^n \right| \leq \sum_{n=m+1}^{\infty} r^n \xrightarrow[m \to \infty]{} 0$$

Corollary: For f in the theorem, we have $\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \, dw$

Application: $$\int_{|w|=r} \frac{\overbrace{\sin(e^w)}^{= f(w)}}{w^2} \, dw = 2\pi i f'(0)$$

Theorem: Let $f: \mathbb{C} \to \mathbb{C}$, analytic (holomorphic) (complex differentiable everywhere)
called entire. If f is bounded ($\sup_{z\in\mathbb{C}} |f(z)| < \infty$). then f is constant

$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$

$x \in \mathbb{R}$ $\sin(ix) = \frac{1}{2i}(e^{-x} - e^{x})$

Proof. $f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|w|=r} \frac{f(w)}{(w-z)^{n+1}} dw$

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \left| \int_{|w|=r} \frac{f(w)}{(w-z)^{n+1}} dw \right| \leq \frac{n!}{2\pi} \sup_{|w|=r} \frac{|f(w)|}{|w-z|^{n+1}} 2\pi r$$

$$= r\, n! \sup_{|w|=r} \frac{|f(w)|}{|w-z|^{n+1}}$$

$\Rightarrow |f^{(n)}(0)| = \frac{n!}{r^n} \sup_{|w|=r} |f(w)| \xrightarrow[r\to\infty]{} 0$ for $n \geq 1$, since f is bounded.

$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f(0)$. constant

Fundamental Theorem of Algebra:

Theorem: Every non-constant polynomial has a root in $\mathbb{C}$.

Proof. $p(z) = a_0 + a_1 z + \cdots + a_n z^n \leftarrow$ entire. Assume $p(z) \neq 0$ for $\forall z \in \mathbb{C}$

$q(z) = \frac{1}{p(z)} \leftarrow$ is also entire. q is also bounded since

$\lim_{|z|\to\infty} p(z) = \infty \Rightarrow \lim_{|z|\to\infty} \frac{1}{p(z)} = 0$. Fix $\varepsilon > 0$, then $\exists$ $R > 0$. $|z| \geq R$, $\frac{1}{|q(z)|} < \varepsilon$.

Observe that $|z| \leq R$ is a compact set.

Recall $q(z) = \frac{1}{p(z)}$ is CTS and thus $|q(z)| \leq r'$ for $\forall |r| \leq R$. $\Rightarrow q$ is
constant. (L... Theorem)

Theorem: Let $f: D \to \mathbb{C}$ be analytic and D is simply connected.

Then f has an antiderivative $F: D \to \mathbb{C}$ s.t. $F' = f$.

$f(z) = z$, $F(z) = \frac{z^2}{2} + C$

Fix $z_0 \in D$, then take any curve $\gamma\ [0,1] \to D$ s.t. $\gamma(0) = z_0$, $\gamma(1) = z$.

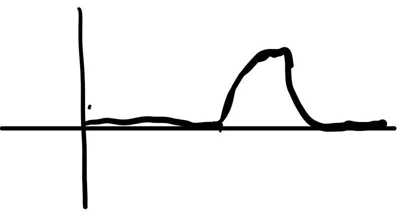
$F(z) = \int_\gamma f(w)\,dw$.

curve independent. $x = y \Rightarrow f(x) = f(y)$.

Proof. The F is well-defined. let $\gamma_1, \gamma_2 : [0,1] \to D$ be two curves starting at z_0. $\gamma_1(0) = \gamma_2(0) = z_0$ and connecting it to z: $\gamma_1(1) = \gamma_2(1) = z$.

In order for F to be well-defined $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.

since $\int_{\gamma_1} f(z)\,dz + \int_{-\gamma_2} f(z)\,dz = 0$.



$$f(x) = \begin{cases} e^{\frac{1}{1-x}}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

infinitely differentiable.

Theorem: Let $f, g: D \to \mathbb{C}$ be analytic. then the following are equivalent

(i). $f = g$

(ii) $\exists z_0 \in D$. $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \geq 0$.

(iii). $\exists z_0 \in D$ and sequence $z_n \to z_0$ $f(z_n) = g(z_n)$

Proof (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii)

$h = f - g \Rightarrow$ If $h^{(n)}(z_0) = 0$ for all $n \geq 0$. $h(z) = \sum_{n=0}^{\infty} \frac{\overbrace{h^{(n)}(z_0)}^{=0}}{n!}(z - z_0)^n = 0$.

This shows (ii) $\Rightarrow$ (i).

Now (iii) $\Rightarrow$ (i).

$h(z_n) = 0$ and since $z_n \to z_0 \Rightarrow h(z_n) \to h(z_0) = 0$.

$h(z)=\sum_{n=m} \frac{h^{(n)}(z_0)}{n!}(z-z_0)^n$, $m \geq 1$. We assume m is such that

$h^{(m)}(z_0) \neq 0$

$$h(z)=\underbrace{(z-z_0)^m}_{\neq 0}\underbrace{\sum_{n=0}^{\infty} \frac{h^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n}$$

$h(z_n)=0$ $\qquad$ $g(z_n)=0$. $z_n \to z_0 \Rightarrow g(z_n) \to g(z_0)=0$.

$$0=g(z_0)=\frac{h^{(m)}(z_0)}{m!}$$